Inverse Trig Substitutions

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Trig Identities

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Recall: The following are important trigonometric identities:

1. $\sin^2(\theta) + \cos^2(\theta) = 1$
2. $\tan^2(\theta) + 1 = \sec^2(\theta)$
3. $\cos^2(\theta) = \frac{\cos(2\theta)+1}{2}$

Example

Example Evaluate $\int \frac{1}{1-x^2}\,dx$.

We make the substitution

$$x = \sin(\theta) \Rightarrow \theta = \arcsin(x), dx = \cos(\theta)\,d\theta$$

$$\begin{aligned}
\int \frac{1}{1-x^2}\,dx &= \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\,d\theta \\
&= \int \frac{\cos(\theta)}{\cos^2(\theta)}\,d\theta \\
&= \int \sec(\theta)\,d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + C
\end{aligned}$$

Problem: How do we convert this back to a function of x?

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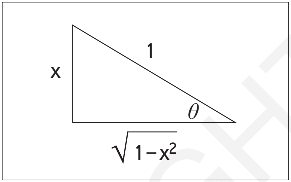
Example



Note: We have $x = \sin(\theta) \Rightarrow \theta = \arcsin(x), -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} \Rightarrow \quad & \cos(\theta) = \sqrt{1-x^2} \\ \Rightarrow \quad & \sec(\theta) = \frac{1}{\sqrt{1-x^2}} \\ \Rightarrow \quad & \tan(\theta) = \frac{x}{\sqrt{1-x^2}} \end{aligned}$$

Therefore

$$\int \frac{1}{1-x^2}\,dx = \ln|\sec(\theta)+\tan(\theta)| + C = \ln\left|\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}\right| + C$$

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Example

Example: Evaluate $\int \frac{1}{(x^2+1)^2}\,dx$.

We make the substitution $\theta = \arctan(x) \Rightarrow x = \tan(\theta), dx = \sec^2(\theta)\,d\theta$.

$$\begin{aligned}
\int \frac{1}{(x^2+1)^2}\,dx &= \int \frac{\sec^2(\theta)}{(\tan^2(\theta)+1)^2}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{(\sec^2(\theta))^2}\,d\theta \\
&= \int \frac{1}{\sec^2(\theta)}\,d\theta \\
&= \int \cos^2(\theta)\,d\theta \\
&= \int \left(\frac{\cos(2\theta)}{2} + \frac{1}{2}\right) d\theta \\
&= \frac{1}{2}\cdot\frac{\sin(2\theta)}{2} + \frac{\theta}{2} + C \\
&= \frac{1}{4}(\sin(2\theta)) + \frac{\theta}{2} + C
\end{aligned}$$

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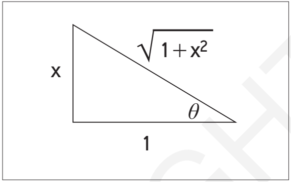
Example



Note: We have $\theta = \arctan(x) \Rightarrow x = \tan(\theta), -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\begin{aligned} \Rightarrow \quad \cos(\theta) &= \frac{1}{\sqrt{1+x^2}} \\ \Rightarrow \quad \sin(\theta) &= \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{4}(\sin(2\theta)) + \frac{\theta}{2} + C &= \frac{1}{2}(\sin(\theta)\cos(\theta)) + \frac{\theta}{2} + C \\ &= \frac{1}{2}(\frac{x}{1+x^2} + \arctan(x)) + C \end{aligned}$$

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Example

Example: Evaluate $\int_0^1 \frac{1}{(x^2+1)^2}\, dx$.

We make the substitution $\theta = \arctan(x) \Rightarrow x = \tan(\theta)$ to get

$$\begin{aligned}
\int_0^1 \frac{1}{(x^2+1)^2}\, dx &= \int_{\arctan(0)}^{\arctan(1)} \frac{\sec^2(\theta)}{(\tan^2(\theta)+1)^2}\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \cos^2(\theta)\, d\theta \\
&= \frac{1}{4}\,(\sin(2\theta)) + \frac{\theta}{2}\Big|_0^{\frac{\pi}{4}} \\
&= \frac{1}{4} + \frac{\pi}{8}
\end{aligned}$$

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Summary

Expression	Substitution
$a^2 - x^2$	$x = a\sin(\theta), -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$a^2 + x^2$	$x = a\tan(\theta), -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$x^2 - a^2$	$x = a\sec(\theta), 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$

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