Integration by Parts

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The Integration by Parts Formula

Recall: The Product Rule for derivatives states that

$$\begin{aligned} \frac{d}{dx}(f \cdot g)(x) &= \left(\frac{df}{dx}\right)(x) \cdot g(x) + f(x) \cdot \left(\frac{dg}{dx}\right)(x) \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

Key Observation: To *undo* the Product Rule we get that if f and g are differentiable,

$$f(x)g(x) = \int \frac{d}{dx}\left(f(x)g(x)\right)\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Definition: [Integration by Parts Formula]

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

or

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

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The Strategy

Example: Evaluate $\int xe^x \, dx$.

Strategy:

- The goal is to rid the integrand of the x.
- The method to remove the x is to differentiate it "downwards" to produce the constant 1.
- As a compensation, we must integrate the other factor e^x "upwards".
- Integrating e^x is not difficult.

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The Strategy

Example: Evaluate $\int xe^x\,dx$.

Solution: Let

$$f(x) = x \text{ and } g'(x) = e^x$$

Then

$$f'(x) = 1 \text{ and } g(x) = e^x$$

Integrating by parts, we have

$$\begin{aligned} \int xe^x\,dx = \int f(x)g'(x)\,dx &= f(x)g(x) - \int f'(x)g(x)\,dx \\ &= xe^x - \int 1 \cdot e^x\,dx \\ &= xe^x - e^x + C \end{aligned}$$

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More Examples

Example: Evaluate $\int x^2 \sin(x)\, dx$.

Strategy: We would like to eliminate the x^2 term. We can do this by differentiating twice $\Rightarrow$ using Integration by Parts twice!

Step 1: Let $f(x) = x^2$, $g'(x) = \sin(x)$. Then $f'(x) = 2x$ and $g(x) = -\cos(x)$.

$$\begin{aligned} \int x^2 \sin(x)\, dx &= -x^2 \cos(x) - \int 2x(-\cos(x))\, dx \\ &= -x^2 \cos(x) + 2 \int x \cos(x)\, dx \end{aligned}$$

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More Examples

Example (continued): Evaluate $\int x^2 \sin(x)\, dx$.

Step 2: Evaluate $\int x \cos(x)\, dx$.
Let $f(x) = x$, $g'(x) = \cos(x)$. Then $f'(x) = 1$ and $g(x) = \sin(x)$.

$$\begin{aligned}\int x \cos(x)\, dx &= x \sin(x) - \int \sin(x)\, dx \\ &= x \sin(x) + \cos(x) + C\end{aligned}$$

Therefore

$$\begin{aligned}\int x^2 \sin(x)\, dx &= -x^2 \cos(x) + 2 \int x \cos(x)\, dx \\ &= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C\end{aligned}$$

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