Comparison Test for Integrals

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Monotone Convergence Theorem for Functions

Theorem: [Monotone Convergence Theorem for Functions (MCTF)]

Assume that F is nondecreasing on $[a, \infty)$. Let

$$S = \{F(x) \mid x \in [a, \infty)\}$$

1) If S is bounded above, then $\lim_{x \to \infty} F(x) = L = lub(S)$.
2) If S is not bounded above, then $\lim_{x \to \infty} F(x) = \infty$.

Application to Improper Integrals:

Assume that f is continuous and positive on $[a, \infty)$. For each $b \in [a, \infty)$ define

$$F(b) = \int_a^b f(t)\, dt.$$

Then $\int_a^\infty f(t)\, dt$ converges if and only if

$$S = \{F(b) \mid b \in [a, \infty)\}$$

is bounded above. In case of convergence

$$\int_a^\infty f(t)\, dt = L = lub(S)$$

Comparison Test

Question: We know from the p-Test that

$$\int_1^\infty \frac{1}{x^4}\,dx$$

converges. What can we say about

$$\int_1^\infty \frac{1}{1+x^4}\,dx\ ?$$

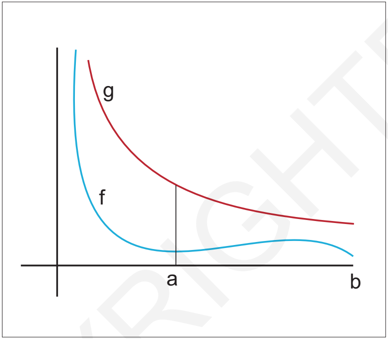
Key Observation: We know that

$$0 < \frac{1}{1+x^4} < \frac{1}{x^4}$$

for all $x \geq 1$ so is it true that

$$\int_1^\infty \frac{1}{x^4}\,dx < \infty \Rightarrow \int_1^\infty \frac{1}{1+x^4}\,dx < \infty\ ?$$

Comparison Test

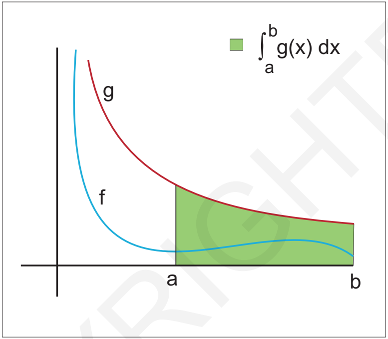


Theorem: [Comparison Test for Improper Integals]

Assume that $0 \leq f(x) \leq g(x)$ for all $x \geq a$ and that both f and g are integrable on $[a, b]$ for all $b > a$.

1. If $\int_a^\infty g(x)\, dx$ converges, then so does $\int_a^\infty f(x)\, dx$.

Comparison Test



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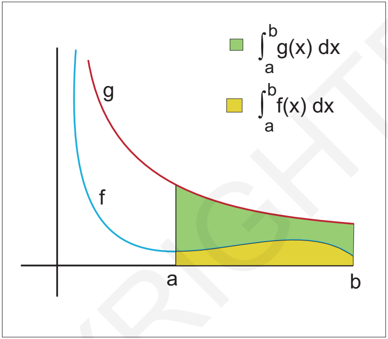

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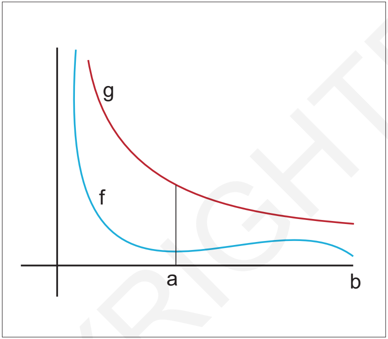

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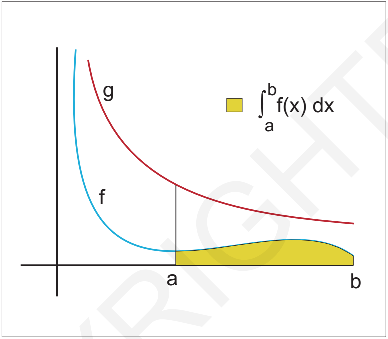

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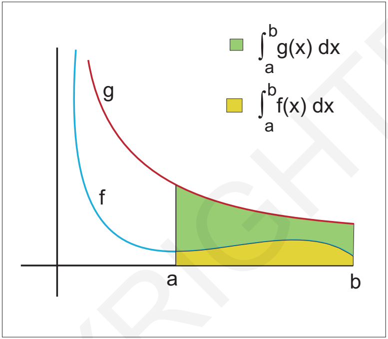

Theorem: [Comparison Test for Improper Integals]

Assume that $0 \leq f(x) \leq g(x)$ for all $x \geq a$ and that both f and g are integrable on $[a, b]$ for all $b > a$.

1. If $\int_a^\infty g(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$.
2. If $\int_a^\infty f(x)\,dx$ diverges, then so does $\int_a^\infty g(x)\,dx$.

Comparison Test

Proof:

1) Assume that $\int_a^\infty g(x)\,dx$ converges. For each $b \in [a, \infty)$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \le \int_a^\infty g(x)\,dx.$$

By the Monotone Convergence Theorem for Functions $\int_a^\infty f(x)\,dx$ converges.

2) Assume that $\int_a^\infty f(x)\,dx$ diverges. Then

$$\lim_{b\to\infty} \int_a^b f(x)\,dx = \infty$$

However, since

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

for each $b > a$, we have

$$\lim_{b\to\infty} \int_a^b g(x)\,dx = \infty$$

Comparison Test

Example: We know that

$$0 < \frac{1}{1+x^4} < \frac{1}{x^4}$$

for all $x \geq 1$ and so $\int_1^\infty \frac{1}{x^4}\,dx$ converges by the p-Test.

It follows that

$$\int_1^\infty \frac{1}{1+x^4}\,dx$$

converges by The Comparison Test.

Comparison Test

Problem: Does

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converge?

Observation : We know that

$$\frac{\sin(x)}{x^2} \leq \frac{1}{x^2}$$

for all $x > 0$ and that

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges.

Question: Does this mean that

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converges?

Note: $f(x) = \frac{\sin(x)}{x^2}$ is not positive.

Comparison Test

Problem (continued): Does

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converge?

Observation : We have that

$$\frac{|\sin(x)|}{x^2} \leq \frac{1}{x^2}$$

for all $x > 0$ and hence

$$\int_1^\infty \frac{|\sin(x)|}{x^2}\,dx$$

also converges by the Comparison Test.

Claim: We claim that this shows that

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converges.

Absolute Convergence

Definition: [Absolute Convergence for Type I Improper Integrals]

Let f be integrable on $[a, b]$ for all $b \geq a$. We say that the improper integral

$$\int_a^\infty f(x)\,dx$$

converges absolutely if

$$\int_a^\infty \mid f(x) \mid \, dx$$

converges.

Absolute Convergence

Theorem: [Absolute Convergence Theorem for Improper Integrals]

Let f be integrable on $[a, b]$ for all $b > a$. Then $|f|$ is also integrable on $[a, b]$ for all $b > a$. Moreover, if we assume that

$$\int_a^\infty |\, f(x) \,| \, dx$$

converges, then so does

$$\int_a^\infty f(x)\, dx$$

In particular, if

$$0 \leq |f(x)| \leq g(x)$$

for all $x \geq a$, both f and g are integrable on $[a, b]$ for all $b \geq a$, and if

$$\int_a^\infty g(x)\, dx$$

converges, then so does

$$\int_a^\infty f(x)\, dx$$

Absolute Convergence

Proof: We will not prove the integrability of $|f|$.

Let

$$h(x) = f(x) + |\, f(x) \,|$$

Then

$$0 \leq h(x) \leq 2 \,|\, f(x) \,|$$

so by the Comparison Test

$$\int_a^\infty h(x)\, dx$$

converges.

Therefore, so does

$$\int_a^\infty f(x)\, dx$$

and in fact

$$\int_a^\infty f(x)\, dx = \int_a^\infty h(x) dx - \int_a^\infty |\, f(x) \,|\ dx$$

Absolute Convergence

Example: Show that

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converges.

We have

$$0 \leq \frac{|\sin(x)|}{x^2} \leq \frac{1}{x^2}$$

so by the Comparison Test and the p-Test.

$$\int_1^\infty \frac{|\sin(x)|}{x^2}\,dx$$

converges. It follows that

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converges.