Differentiation of an Integral Function

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Integral Functions

Remark:

Assume that the function f is continuous on an interval $[a, b]$. Define the *integral function* G on $[a, b]$ by

$$G(x) = \int_a^x f(t)\,dt.$$

Questions:

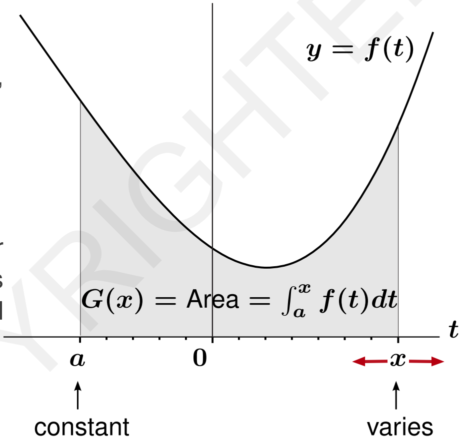
What are the properties of G?
Is it continuous?
Is it differentiable?
If so what is $G'(x)$?

Integral Functions

Note: If $f \geq 0$ on $[a, b]$, then

$$G(x) = \int_a^x f(t)\,dt$$

calculates the area under the graph of $y = f(t)$ as x *varies* over an interval $[a, b]$ starting from a.



Fundamental Question: Is G differentiable and what is $G'(x)$?

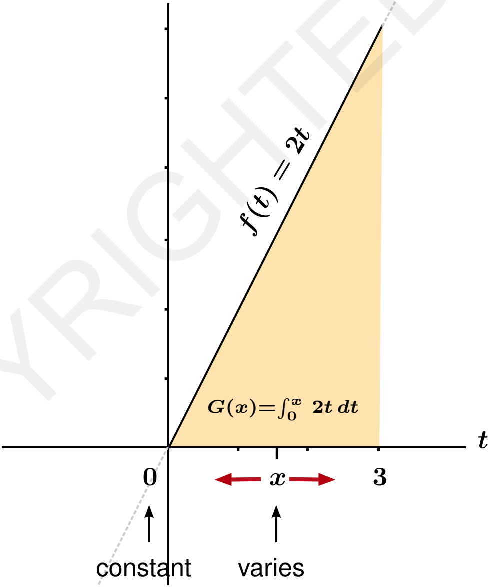
Integral Functions: An Example

Example:

Let $f(t) = 2t$ on $[0, 3]$.

Find a formula for

$$G(x) = \int_0^x 2t\,dt.$$



Integral Functions: An Example

Case 1:

If $x = 0$, we have that

$$G(0) = \int_0^0 2t\,dt = 0$$

since the limits of integration are identical. (There is no area to calculate.)

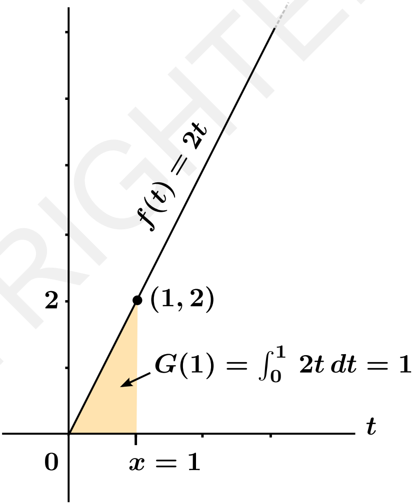
Thus we have the area under $f(t)$ on the interval $[0, 0]$ is 0 and

$$G(0) = 0.$$

Integral Functions: An Example

Case 2:

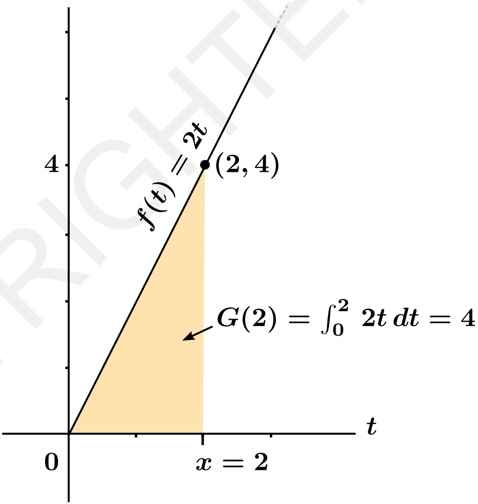
$$\begin{aligned} \text{Area} &= G(1) \\ &= \int_0^1 2t\,dt \\ &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2}(1)(2(1)) \\ &= 1 \end{aligned}$$



Integral Functions: An Example

Case 3:

$$\begin{aligned} \text{Area} &= G(2) \\ &= \int_0^2 2t\,dt \\ &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2}(2)(2(2)) \\ &= 4 \end{aligned}$$



Integral Functions: An Example

Case 4:

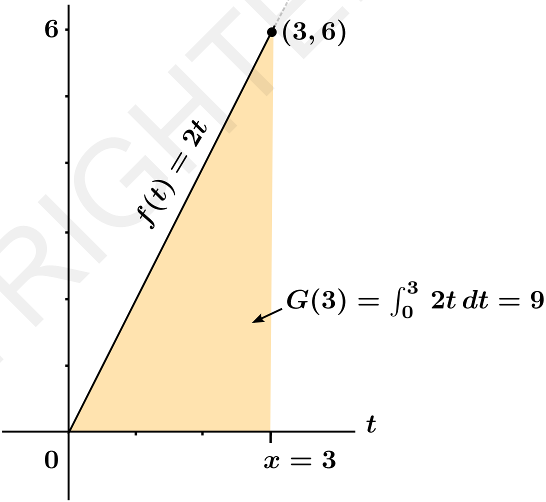
$$\text{Area} = G(3)$$

$$= \int_0^3 2t\,dt$$

$$= \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2}(3)(2(3))$$

$$= 9$$



Integral Functions: An Example

x	$G(x)$
1	1
2	4
3	9
$\vdots$	$\vdots$
x	x^2

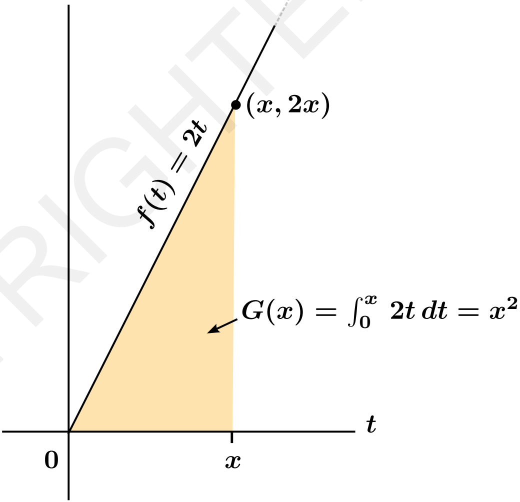
Key Observation: The pattern suggests that

$$G(x) = \int_0^x 2t\,dt = x^2$$

Integral Functions: An Example

Case 5:

$$\text{Area} = G(x) = \int_0^x 2t\,dt = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(x)(2(x)) = x^2$$



Integral Functions: An Example

Important Observation:

Notice that if $f(t) = 2t$ on $[0, 3]$ and if

$$G(x) = \int_0^x f(t)\,dt,$$

then

$$G(x) = x^2$$

and the derivative of G is

$$G'(x) = 2x.$$

This means that

$$G'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x).$$