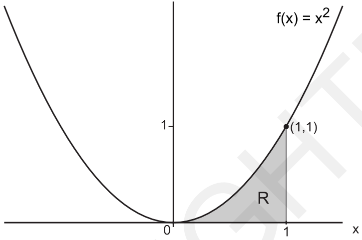
Geometric Interpretation of the Integral

Created by

Barbara Forrest and Brian Forrest

Geometric Interpretation of the Integral



Remark: We have already seen that the area of the region R bounded by the graph of $f(x) = x^2$, by the x-axis, and by the lines $x = 0$ and $x = 1$ is

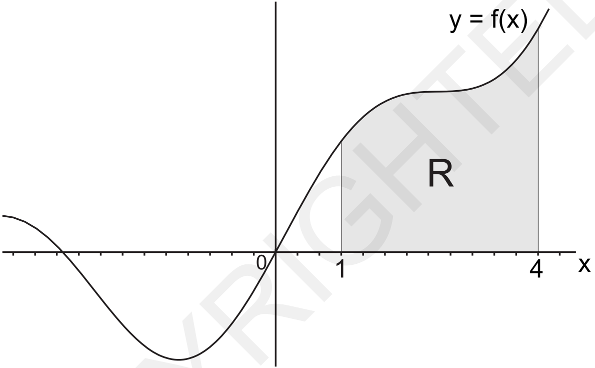
$$\int_0^1 x^2 \, dx$$

In fact, whenever $\mathbf{f(x)} \geq 0$ on all of $[a, b]$, the area under the graph of f and above the x-axis bounded by the lines $x = a$ and $x = b$ will be

$$\int_a^b f(x) \, dx$$

Question: What happens if $f(x) \leq 0$ on some part of $[a, b]$?

Geometric Interpretation of the Integral

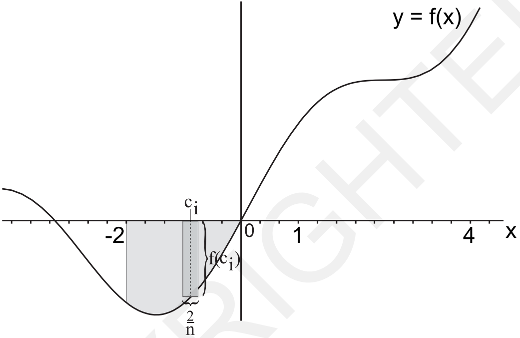


Example: Assume that f is as shown in the diagram. Then $\int_1^4 f(x)\,dx$ is the area of the region R bounded by the graph of f, the x-axis, and the lines $x = 1$ and $x = 4$.

Question: Suppose instead that we wanted to calculate $\int_{-2}^{0} f(x)\,dx$.

What would this represent?

Geometric Interpretation of the Integral



Observation: Consider a term

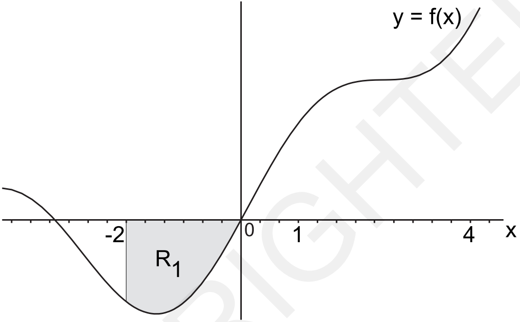
$$f(c_i)\frac{2}{n}$$

in a generic Riemann sum. Since $f(c_i)$ is the **negative** of the height of the pictured rectangle

$$S_n = \sum_{i=1}^{n} f(c_i)\frac{2}{n}$$

approximates the **negative** of the area bounded by the graph of f, the x-axis, and the lines $x = -2$ and $x = 0$.

Geometric Interpretation of the Integral



Observation (continued): Since

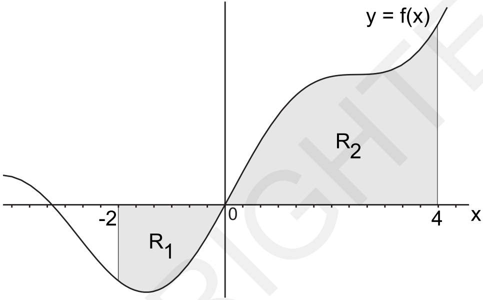
$$S_n = \sum_{i=1}^{n} f(c_i)\frac{2}{n}$$

approximates the **negative** of the area bounded by the graph of f, the x-axis, and the lines $x = -2$ and $x = 0$,

$$\int_{-2}^{0} f(x)\ dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{2}{n}$$

is the **negative** of the area of the region R_1.

Geometric Interpretation of the Integral



We have

$$\int_{-2}^{4} f(x)\,dx = \int_{-2}^{0} f(x)dx + \int_{0}^{4} f(x)dx$$

$$= R_2 - R_1$$

Geometric Interpretation of the Integral

Geometric Interpretation of the Integral

If f is a continuous function on the interval $[a, b]$, then

$$\int_a^b f(x)\, dx$$

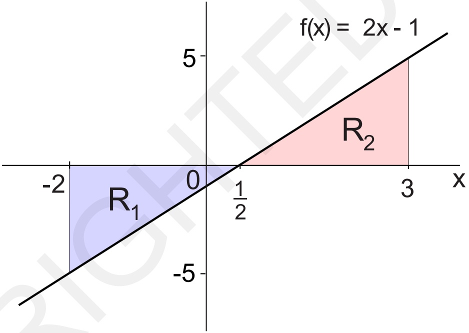
represents the area of the region *under* the graph of f that lies *above* the x-axis between $x = a$ and $x = b$ ***minus*** the area of the region *above* the graph of f that lies *below* the x-axis between $x = a$ and $x = b$.

Geometric Interpretation of the Integral

Example: Evaluate

$$\int_{-2}^{3} (2x - 1)\ dx$$

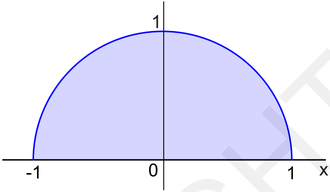
geometrically as the difference in the area of two triangles.



Solution:

$$\begin{aligned}\int_{-2}^{3} (2x - 1)\ dx &= \int_{-2}^{\frac{1}{2}} (2x - 1) dx + \int_{\frac{1}{2}}^{3} (2x - 1) dx \\ &= -R_1 + R_2 \\ &= -\frac{5 \times (\frac{1}{2} - (-2))}{2} + \frac{5 \times (3 - \frac{1}{2})}{2} \\ &= -\frac{25}{4} + \frac{25}{4} = 0\end{aligned}$$

Geometric Interpretation of the Integral



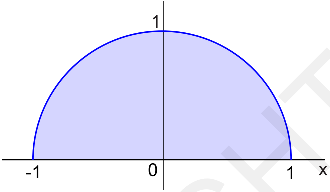
Example: Evaluate

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

Solution: The shape of this region is a semi-circle with radius 1.
It follows that

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \frac{\pi}{2}.$$

Geometric Interpretation of the Integral



Example: Evaluate

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

Solution: The shape of this region is a semi-circle with radius 1.
It follows that

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$

Question: How would we evaluate

$$\int_{-1}^{\frac{1}{2}} \sqrt{1-x^2}\,dx \qquad ?$$