Fundamental Theorem of Calculus (Part 2)

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Fundamental Theorem of Calculus (Part 2)

Example: Evaluate

$$\int_0^2 t^3\,dt.$$

Observation: If

$$G(x) = \int_0^x t^3\,dt,$$

then

$$G(2) = \int_0^2 t^3\,dt.$$

We know from the FTC1 that

$$G'(x) = x^3$$

so G is an antiderivative of x^3. Hence there exists a constant C such that

$$G(x) = \int_0^x t^3\,dt = \frac{x^4}{4} + C.$$

Question: How does this help us?

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Note: We have just seen that

$$G(x) = \int_0^x t^3\,dt = \frac{x^4}{4} + C$$

for some constant $C \in \mathbb{R}$.

However, we also know that

$$0 = \int_0^0 t^3\,dt = G(0) = \frac{0^4}{4} + C = C$$

so

$$G(x) = \int_0^x t^3\,dt = \frac{x^4}{4}.$$

Finally,

$$\int_0^2 t^3\,dt = G(2) = \frac{2^4}{4} = 4.$$

Question: Did we really need to find C?

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Key Observation:

Let F and G be any two antiderivatives of the same function f. Then

$$G(x) = F(x) + C.$$

Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} G(b) - G(a) &= (F(b) + C) - (F(a) + C) \\ &= F(b) - F(a). \end{aligned}$$

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Key Observation (continued):

Assume that f is continuous and that we want to know

$$\int_a^b f(t)\,dt.$$

Let

$$G(x) = \int_a^x f(t)\,dt.$$

Then G is an antiderivative of f and if F is any other antiderivative of f, then

$$\begin{aligned}\int_a^b f(t)\,dt &= G(b) \\ &= G(b) - G(a) \quad \text{(since } G(a) = 0\text{)} \\ &= F(b) - F(a)\end{aligned}$$

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Example: Evaluate

$$\int_0^2 t^3\,dt.$$

Solution: We know that

$$F(x) = \frac{x^4}{4}$$

is an antiderivative for $f(x) = x^3$.

Hence

$$\begin{aligned} \int_0^2 t^3\,dt &= F(2) - F(0) \\ &= \frac{2^4}{4} - \frac{0^4}{4} \\ &= 4 \end{aligned}$$

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Theorem: [Fundamental Theorem of Calculus (Part 2) [FTC2]]

Assume that f is continuous and that F is any antiderivative of f. Then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Notation: We write

$$F(x)\Big|_a^b = F(b) - F(a)$$

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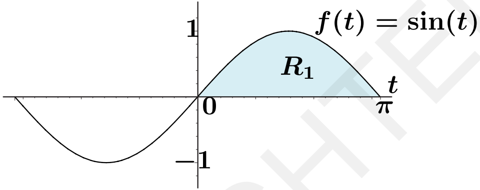

Example: Evaluate

$$\int_0^{\pi} \sin(t)\,dt.$$

Solution: Since $f(t) = \sin(t)$ is continuous and $F(t) = -\cos(t)$ is an antiderivative of f, by the FTC2, we have

$$\begin{aligned}\int_0^{\pi} \sin(t)\,dt &= -\cos(t)\Big|_0^{\pi} \\ &= -\cos(\pi) - (-\cos(0)) \\ &= 2\end{aligned}$$

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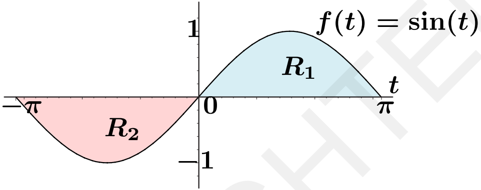

Example: Evaluate

$$\int_{-\pi}^{\pi} \sin(t)dt$$

Solution: Using a geometrical argument, this integral should equal 0.

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(t)\,dt &= -\cos(t)\Big|_{-\pi}^{\pi} \\ &= -\cos(\pi) - (-\cos(-\pi)) \\ &= 0 \end{aligned}$$

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Key Remark: It is important that we emphasize the difference between the meaning of

$$\int_a^b f(t)\,dt \qquad \text{and} \qquad \int f(t)dt$$

The first expression,

$$\int_a^b f(t)\,dt$$

is called a *definite* integral. It represents a *number* that is defined as a limit of Riemann sums.

The second expression,

$$\int f(t)dt$$

is called an ***in**definite* integral. It represents the *family of all functions that are antiderivatives* of the given function f.