Fundamental Theorem of Calculus (Part 1): Examples

Created by

Barbara Forrest and Brian Forrest

Fundamental Theorem of Calculus (Part 1)

Theorem: [Fundamental Theorem of Calculus (Part 1) [FTC1]]

Assume that f is continuous on an open interval I containing a point a. Let

$$G(x) = \int_a^x f(t)\,dt.$$

Then $G(x)$ is differentiable at each $x \in I$ and

$$G'(x) = f(x).$$

Equivalently,

$$G'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Fundamental Theorem of Calculus (Part 1)

Example: Find $F'(x)$ if

$$F(x) = \int_3^x e^{t^2}\,dt.$$

Solution: Since $f(t) = e^{t^2}$ is a continuous function, the Fundamental Theorem of Calculus 1 tells us we can replace t by x in $f(t)$ to get

$$F'(x) = e^{x^2}.$$

Question: What happens if we modify the previous question with

$$G(x) = \int_3^{x^2} e^{t^2}\,dt?$$

Fundamental Theorem of Calculus (Part 1)

Example: Find $G'(x)$ if

$$G(x) = \int_3^{x^2} e^{t^2}\, dt.$$

Solution: Let

$$H(u) = \int_3^{u} e^{t^2}\, dt.$$

Then

$$H'(u) = e^{u^2}.$$

However,

$$G(x) = H(x^2)$$

so the Chain Rule tells us that

$$G'(x) = H'(x^2) \cdot \frac{d}{dx}\, x^2 = e^{(x^2)^2} \cdot (2x) = 2xe^{x^4}.$$

Fundamental Theorem of Calculus (Part 1)

Remark: Assume that f is continuous and that $u = g(x)$ is differentiable. Let

$$G(x) = \int_a^{g(x)} f(t)\,dt.$$

If

$$H(u) = \int_a^{u} f(t)\,dt,$$

then

$$G(x) = H(g(x))$$

so by the Chain Rule and the FTC1 we get

$$G'(x) = H'(g(x))g'(x) = f(g(x))g'(x).$$

Fundamental Theorem of Calculus (Part 1)

Example: Find $G'(x)$ if

$$G(x) = \int_{\cos(x)}^{a} e^{t^2}\, dt.$$

Note: The bottom limit now varies!

Key Observation:

$$\begin{aligned}
G'(x) &= \frac{d}{dx}\int_{\cos(x)}^{a} e^{t^2}\, dt \\
&= \frac{d}{dx}\Big(-\int_{a}^{\cos(x)} e^{t^2}\, dt\Big) \\
&= -\Big(\frac{d}{dx}\int_{a}^{\cos(x)} e^{t^2}\, dt\Big) \\
&= -\Big(e^{\cos^2(x)} \cdot (-\sin(x))\Big) \\
&= \sin(x) e^{\cos^2(x)}
\end{aligned}$$

Fundamental Theorem of Calculus (Part 1)

Example: Find $H'(x)$ if

$$H(x) = \int_{\cos(x)}^{x^2} e^{t^2}\,dt.$$

Key Observation:

$$\begin{aligned} H(x) &= \int_{\cos(x)}^{3} e^{t^2}\,dt + \int_{3}^{x^2} e^{t^2}\,dt \\ &= \int_{3}^{x^2} e^{t^2}\,dt - \int_{3}^{\cos(x)} e^{t^2}\,dt \end{aligned}$$

Hence

$$\begin{aligned} H'(x) &= \frac{d}{dx}\int_{3}^{x^2} e^{t^2}\,dt - \frac{d}{dx}\int_{3}^{\cos(x)} e^{t^2}\,dt \\ &= 2xe^{x^4} - (-\sin(x))e^{\cos^2(x)} \end{aligned}$$

Fundamental Theorem of Calculus (Part 1)

Theorem:
[Extended Version of the Fundamental Theorem of Calculus]

Assume that f is continuous and that g and h are differentiable. Let

$$H(x) = \int_{g(x)}^{h(x)} f(t)\,dt.$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(x))g'(x).$$