Fundamental Theorem of Calculus (Part 1)

Created by

Barbara Forrest and Brian Forrest

Differentiation of Integral Functions

Important Observation: Notice that if $f(t) = 2t$ on $[0, 3]$ and if

$$G(x) = \int_0^x f(t)\,dt,$$

then

$$G(x) = x^2$$

and the derivative of G is

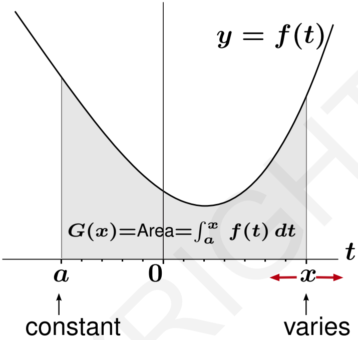
$$G'(x) = 2x.$$

This means that

$$G'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x) \quad (*)$$

Fundamental Question: Does $(*)$ hold for all integral functions?

Differentiation of Integral Functions

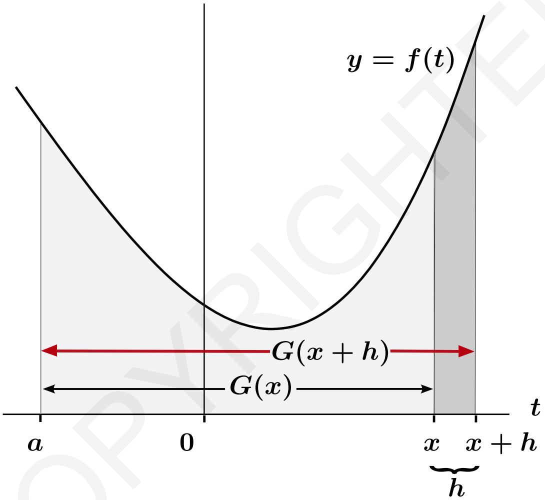


Special Case: Assume that $f(t) \geq 0$ and that f is continuous on the interval $[a, b]$ and let the integral function be defined by

$$G(x) = \int_a^x f(t)\,dt.$$

In this case, $G(x)$ represents the area bounded by the graph of f, the t-axis, and the lines $t = a$ and $t = x$.

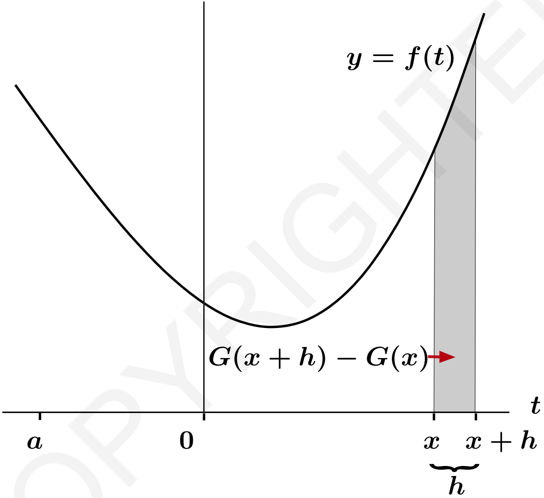
Differentiation of Integral Functions



Note: For a small $h > 0$

$$G(x+h) - G(x) =$$

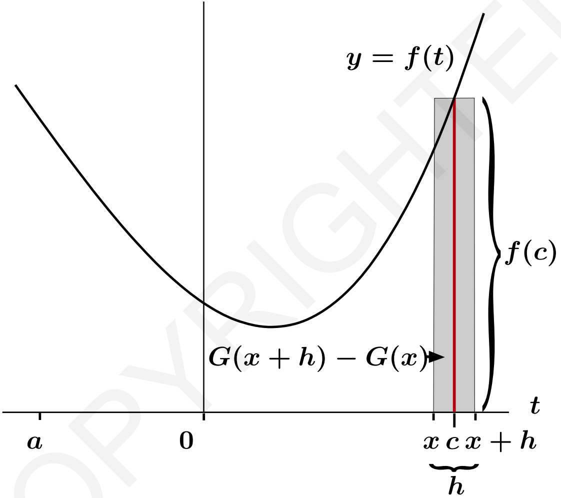
Differentiation of Integral Functions



Note: For a small $h > 0$

$$G(x+h) - G(x) = \int_x^{x+h} f(t)\,dt =$$

Differentiation of Integral Functions



Note: For a small $h > 0$

$$G(x+h) - G(x) = \int_x^{x+h} f(t)\,dt = f(c)h$$

Differentiation of Integral Functions

Remark: When $h > 0$ is small there exists a c with $x < c < x + h$ such that

$$G(x + h) - G(x) = f(c)h$$

and hence

$$\frac{G(x + h) - G(x)}{h} = f(c).$$

However, if h is very small, then c must also be very close to x. Since f is continuous, this means that

$$f(c) = \frac{G(x + h) - G(x)}{h}$$

must be very close to $f(x)$.

We get

$$\lim_{h \to 0^+} \frac{G(x + h) - G(x)}{h} = \lim_{c \to x^+} f(c) = f(x).$$

Fundamental Theorem of Calculus (Part 1)

Theorem: [Fundamental Theorem of Calculus (Part 1) [FTC1]]

Assume that f is continuous on an open interval I containing a point a. Let

$$G(x) = \int_a^x f(t)\,dt.$$

Then $G(x)$ is differentiable at each $x \in I$ and

$$G'(x) = f(x).$$

Equivalently,

$$G'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

Fundamental Theorem of Calculus (Part 1)

Proof of the FTC1: Assume that

$$G(x) = \int_a^x f(t)dt$$

and that f is continuous at $x_0 \in I$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ so that if $0 < |c - x_0| < \delta$, then

$$|f(c) - f(x_0)| < \epsilon.$$

If $0 < |x - x_0| < \delta$, then

$$\begin{aligned} \frac{G(x) - G(x_0)}{x - x_0} &= \frac{\int_a^x f(t)dt - \int_a^{x_0} f(t)dt}{x - x_0} \\ &= \frac{1}{x - x_0} \int_{x_0}^x f(t)dt \end{aligned}$$

By the Average Value Theorem, there exists a c between x and x_0 with

$$f(c) = \frac{1}{x - x_0} \int_{x_0}^x f(t)dt.$$

Fundamental Theorem of Calculus (Part 1)

Proof of the FTC1 (continued):

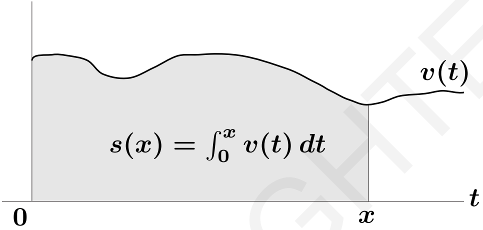
If $0 < |x - x_0| < \delta$, then since $0 < |c - x_0| < \delta$,

$$\left| \frac{G(x) - G(x_0)}{x - x_0} - f(x_0) \right| = |\, f(c) - f(x_0)\,| < \epsilon.$$

By the definition of a limit we get that

$$\begin{aligned} G'(x_0) &= \lim_{x \to x_0} \frac{G(x) - G(x_0)}{x - x_0} \\ &= f(x_0). \end{aligned}$$

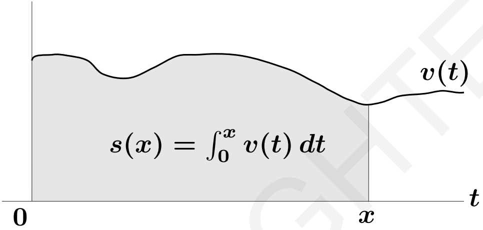
Fundamental Theorem of Calculus (Part 1)



Example: Assume that a vehicle travels forward along a straight road with a velocity at time t given by the function $v(t)$. If we fix a starting point at $t = 0$, then the displacement $s(x)$ up to time $t = x$ is

$$s(x) = \int_0^x v(t)\,dt.$$

Fundamental Theorem of Calculus (Part 1)



Example: Assume that a vehicle travels forward along a straight road with a velocity at time t given by the function $v(t)$. If we fix a starting point at $t = 0$, then the displacement $s(x)$ up to time $t = x$ is

$$s(x) = \int_0^x v(t)\,dt.$$

Since velocity is a continuous function of time, the FTC1 tells us that $s(x)$ is differentiable and that the derivative of displacement is velocity

$$s'(x) = v(x)$$

exactly as we would expect!