Definition of the Integral

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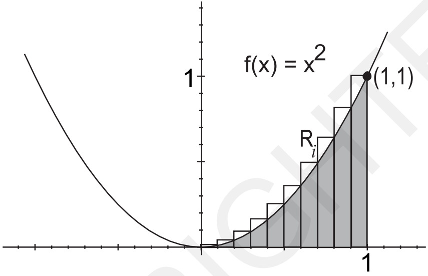
Definition of the Integral

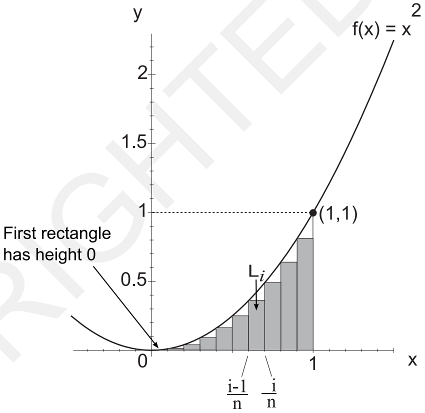


$$
\begin{aligned}
R_n &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3}\frac{(n)(n+1)(2(n)+1)}{6} \\
&= \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6} \to \frac{1}{3}
\end{aligned}
$$

Definition of the Integral

$$L_n = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{2 - \frac{3}{n} + \frac{1}{n^2}}{6} \to \frac{1}{3}$$



If $\{S_n\}$ is any sequence of Riemann sums for $P^{(n)}$, then

$$L_n \leq S_n \leq R_n \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{3}$$

Definition of the Integral

Definition: [Definite Integral]

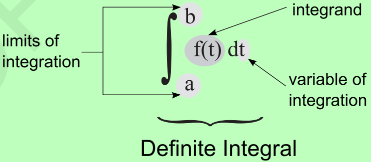
We say that a bounded function f is *integrable* on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty} \|P_n\| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n's, we have

$$\lim_{n\to\infty} S_n = I.$$

In this case, we call I the integral of f over $[a, b]$ and denote it by

$$\int_a^b f(t)\, dt$$

The points a and b are called the *limits of integration* and the function $f(t)$ is called the *integrand*. The variable t is called the *variable of integration*.



Definition of the Integral

Note: The variable of integration is sometimes called a *dummy variable* in the sense that if we were to replace t's by x's everywhere, we would not change the value of the integral.

Question: Are all bounded functions on $[a, b]$ integrable?

Answer: No. Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then f is not integrable on any interval $[a, b]$.

The Integrability Theorem

Theorem: [The Integrability Theorem for Continuous Functions]

Let f be continuous on $[a,b]$. Then f is integrable on $[a,b]$. Moreover,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} S_n$$

where

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

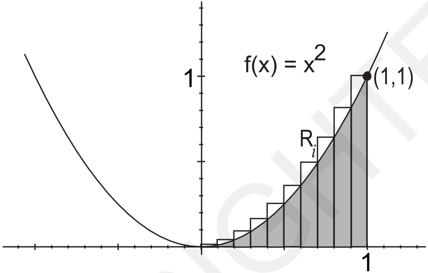
is any Riemann sum associated with the regular n-partitions. In particular,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

and

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}.$$

The Integrability Theorem



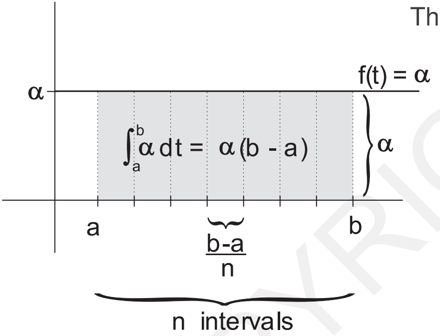
Example: Since

$$R_n = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6}$$

we get

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} = \frac{1}{3}$$

Integrating Constant Functions



Example:
Let $f(t) = \alpha$ for each $t \in [a, b]$. Then

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(t_i)\Delta t_i \\ &= \sum_{i=1}^{n} \alpha\left(\frac{b-a}{n}\right) \\ &= \alpha \sum_{i=1}^{n} \frac{b-a}{n} \\ &= \alpha(b-a) \end{aligned}$$

Since $R_n = \alpha(b-a)$ for each n,

$$\int_a^b \alpha \; dt = \alpha(b-a).$$

Properties of Definite Integrals

Theorem: [Properties of Definite Integrals]

Assume that f and g are integrable on the interval $[a,b]$. Then

i) For any $c \in \mathbb{R}$, $\int_a^b c\, f(t)\; dt = c \int_a^b f(t)\; dt$

ii) $\int_a^b (f+g)(t)\; dt = \int_a^b f(t)\; dt + \int_a^b g(t)\; dt$

iii) If $m \leq f(t) \leq M$ for all $t \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(t)\; dt \leq M(b-a)$$

iv) If $0 \leq f(t)$ for all $t \in [a,b]$, then $0 \leq \int_a^b f(t)\; dt$.

v) If $g(t) \leq f(t)$ for all $t \in [a,b]$, then $\int_a^b g(t)\; dt \leq \int_a^b f(t)\; dt$.

vi) The function $|f|$ is integrable on $[a,b]$ and

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b \mid f(t) \mid dt$$

Properties of Definite Integrals

Proof of (iii): Assume that

$$m \leq f(t) \leq M$$

for all $t \in [a, b]$. Let

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

be any partition of $[a, b]$. Then

$$m(b-a) = \sum_{i=1}^{n} m\Delta t_i \leq \sum_{i=1}^{n} f(t_i)\Delta t_i \leq \sum_{i=1}^{n} M\Delta t_i = M(b-a)$$

since

$$\sum_{i=1}^{n} \Delta t_i = b - a.$$