Change of Variables for the Indefinite Integral

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Remark:

We have seen that antiderivatives are very useful in evaluating definite integrals.

Key Problem:

Are there methods to find antiderivatives of complicated functions?

Antiderivatives and the Chain Rule

Recall: The Chain Rule tells us that if

$$H(x) = f(g(x)),$$

then

$$H'(x) = \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Question: Is there a method to *undo* the Chain Rule?

Change of Variables for the Indefinite Integral

Remark: Assume we have two functions h and f with

$$h'(u) = f(u).$$

Then

$$\int f(u)\,du = h(u) + C.$$

Let $u = g(x)$ be a function of x. The Chain Rule says that if

$$H(x) = h(g(x)),$$

then

$$\begin{aligned} H'(x) &= h'(g(x))g'(x) \\ &= f(g(x))g'(x). \end{aligned}$$

It follows that

$$\begin{aligned} \int f(g(x))g'(x)\,dx &= H(x) + C \\ &= h(g(x)) + C \\ &= h(u) + C \\ &= \int f(u)\,du \end{aligned}$$

Change of Variables for the Indefinite Integral

Definition: [Change of Variables Formula]

Let $u = g(x)$ be differentiable. Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Notation: We will use the notation

$$h(u)\Big|_{u=g(x)}$$

to mean replace u by $g(x)$ in the formula for $h(u)$.

As such the symbol

$$\int f(u)\,du\Big|_{u=g(x)}$$

means replace u by $g(x)$ once the antiderivative has been found.

Method of Substitution

Method of Substitution: Evaluate $\int 2xe^{x^2}\,dx$.

Solution: Let $u = g(x) = x^2$, then $g'(x) = 2x$. If we also let $f(u) = e^u$, then

$$\int 2x\, e^{x^2}\, dx = \int f(g(x))\, g'(x)\, dx$$

We get that

$$\begin{aligned}
\int 2xe^{x^2}\,dx &= \int f(g(x))g'(x)\,dx \\
&= \int f(u)\,du \Big|_{u=g(x)} \\
&= \int e^u\,du \Big|_{u=x^2} \\
&= e^u \Big|_{u=x^2} + C \\
&= e^{x^2} + C
\end{aligned}$$

Method of Substitution

Method of Substitution (continued):

1) Start with
$$\int f(g(x))g'(x)\,dx.$$
2) Identify the possible substitution
$$u = g(x).$$
3) Differentiating both sides gives us
$$\frac{du}{dx} = g'(x).$$
4) If we treat du and dx as if they were "numbers", then
$$du = g'(x)\,dx.$$

We can now *substitute* u for $g(x)$ and du for $g'(x)dx$ to get

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du\,\Big|_{u=g(x)}$$

which is the Change of Variables formula.

Method of Substitution

Example: Evaluate $\int \frac{2x}{1+x^2}\,dx$ by making the substitution $u = 1 + x^2$.

If

$$u = 1 + x^2,$$

then

$$du = 2x\,dx.$$

Substituting $u = 1 + x^2$ and $du = 2x\,dx$ into the original integral gives us

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{u}\,du \Big|_{u=1+x^2}$$

but

$$\int \frac{1}{u}\,du = \ln(|\,u\,|) + C.$$

Hence

$$\begin{aligned} \int \frac{2x}{1+x^2}\,dx &= \int \frac{1}{u}\,du \Big|_{u=1+x^2} \\ &= \ln(|\,u\,|)\Big|_{u=1+x^2} + C \\ &= \ln(|\,1+x^2\,|) + C \end{aligned}$$