Average Value of a Function

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Average Value of a Function

Remark:

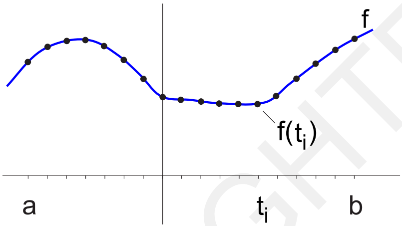
We know that the average of n real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ is

$$\frac{\alpha_1 + \alpha_2 + \ldots + \alpha_n}{n}.$$

Question:

How do you find the average value of a continuous function f on $[a, b]$?

Method 1: Sampling



Method 1: Sampling

One approach would be to take sample values of f and calculate the average of these samples as an estimate of the average value. To obtain the sample, we use the regular n-partition

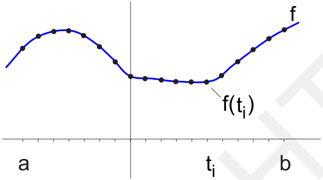
$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

where $t_i = a + \dfrac{i(b-a)}{n}$ and consider

$$\frac{\sum_{i=1}^{n} f(t_i)}{n}$$

To get better estimates we let $n \to \infty$.

Method 1: Sampling



Key Observation:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(t_i)}{n} &= \lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(t_i) \frac{(b-a)}{n} \\
&= \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i) \frac{(b-a)}{n} \\
&= \frac{1}{b-a} \lim_{n\to\infty} R_n \\
&= \frac{1}{b-a} \int_a^b f(t)\, dt
\end{aligned}$$

Method 1: Sampling

Definition: [Average Value of a Function]

If f is continuous on $[a, b]$, the *average value of* f *on* $[a, b]$ is defined as

$$\frac{1}{b-a}\int_a^b f(t)\ dt.$$

Geometric Interpretation of the Average Value

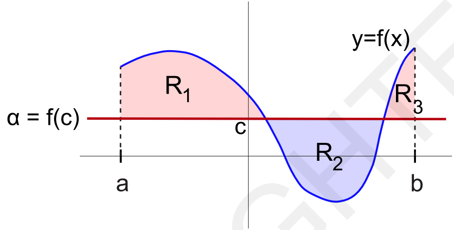
Method 2: Geometric Interpretation

If f is continuous on $[a, b]$, then the *Extreme Value Theorem* implies that there exists $d_1, d_2 \in [a, b]$ with $m = f(d_1)$, $M = f(d_2)$ such that

$$m \leq f(x) \leq M$$

for all $x \in [a, b]$.

Geometric Interpretation of the Average Value



Method 2: Geometric Interpretation (continued)

Since f is continuous and

$$m \le \alpha = \frac{1}{b-a} \int_a^b f(x)\,dx \le M$$

the Intermediate Value Theorem shows that there exists $a < c < b$ so that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Geometrically, it follows that

$$Area\ R_1 + Area\ R_3 = Area\ R_2.$$

Average Value Theorem

Theorem:
[The Average Value Theorem (Mean Value Theorem for Integrals)]

Assume that f is continuous on $[a, b]$. Then there exists $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)\, dt.$$

Average Value Theorem

Key Fact: If $b < a$ and f is continuous on $[b, a]$, then there exists $b \leq c \leq a$ such that

$$\begin{aligned} f(c) &= \frac{1}{a-b}\int_b^a f(t)\,dt \\ &= \frac{1}{a-b}\Big(-\int_a^b f(t)\,dt\Big) \\ &= \frac{1}{b-a}\int_a^b f(t)\,dt \end{aligned}$$

so the Average Value Theorem holds even if $b < a$.

Remark: Average values play a crucial role in probability and statistics.