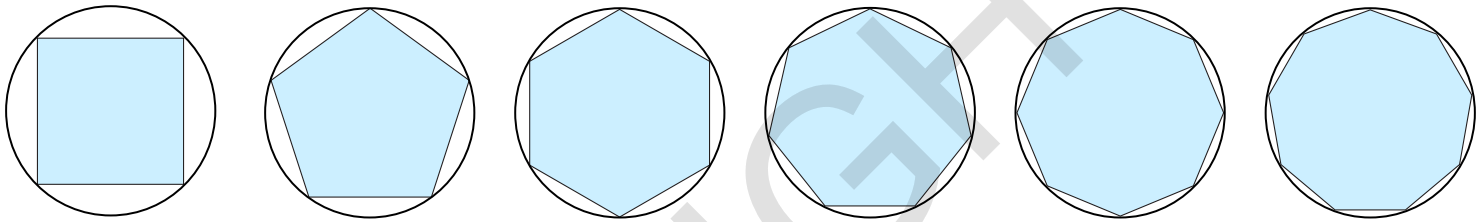
Areas Under Curves

Created by

Barbara Forrest and Brian Forrest

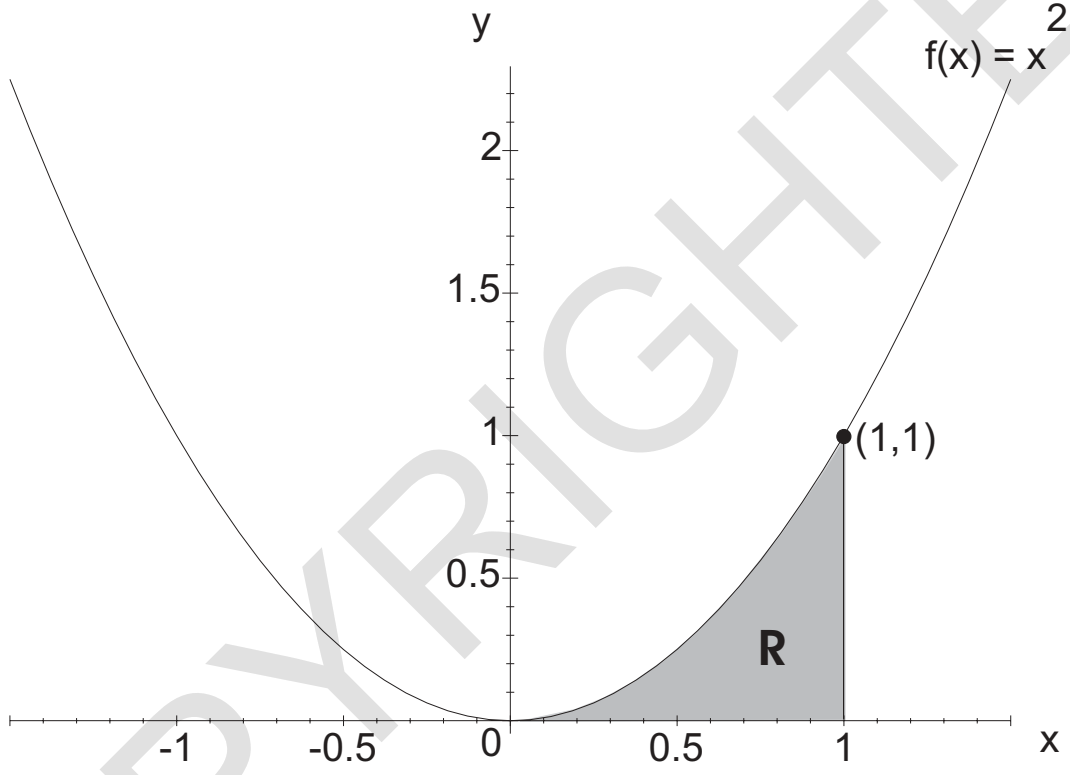
Area of a Circle



Question: How do you find the area of a circle?

Historical Note: Eudoxus of Cnidus and Archimedes used a type of *limit process* know as the the *Method of Exhaustion* to calculate areas.

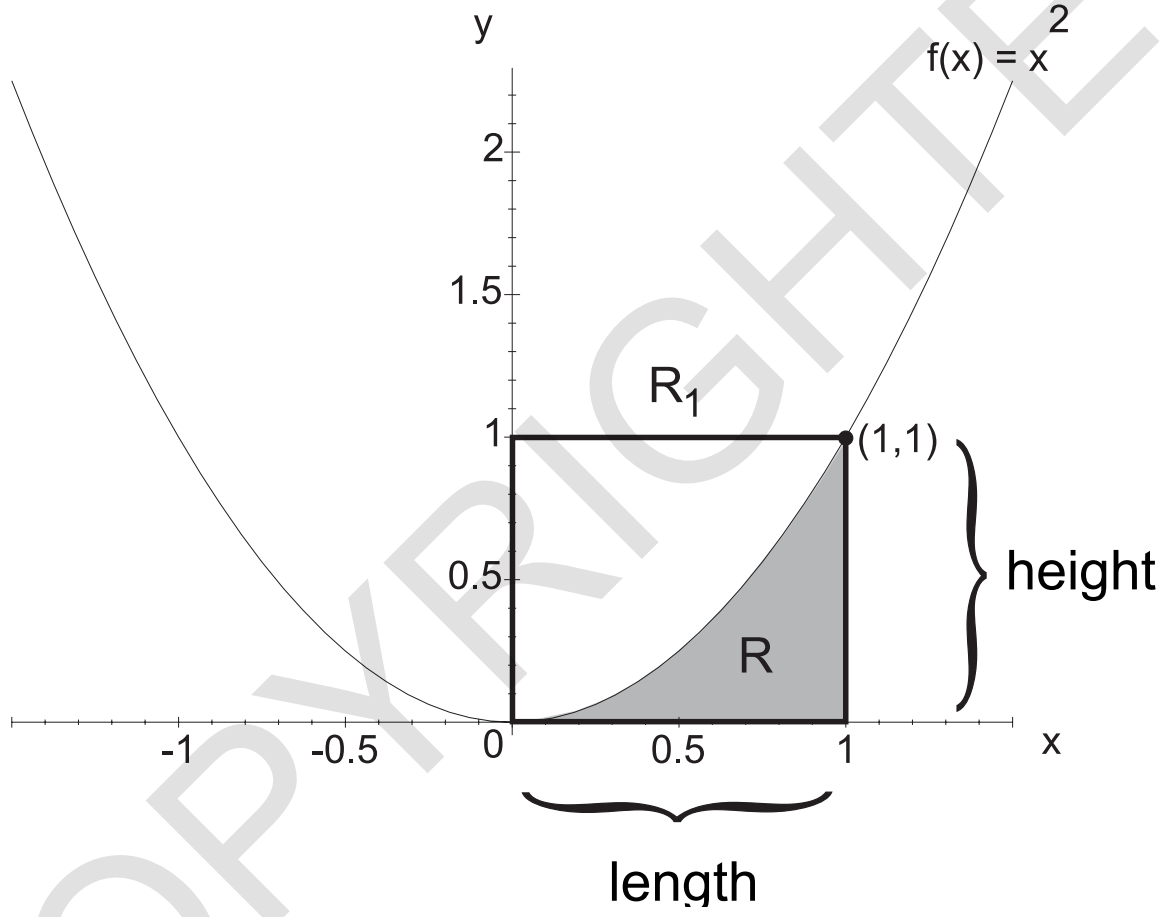
Area Under Curves



Problem: Suppose that $f(x) = x^2$. Consider the region R bounded by the graph of f, by the x-axis, and by the lines $x = 0$ and $x = 1$.

How could we determine the area of this irregular region?

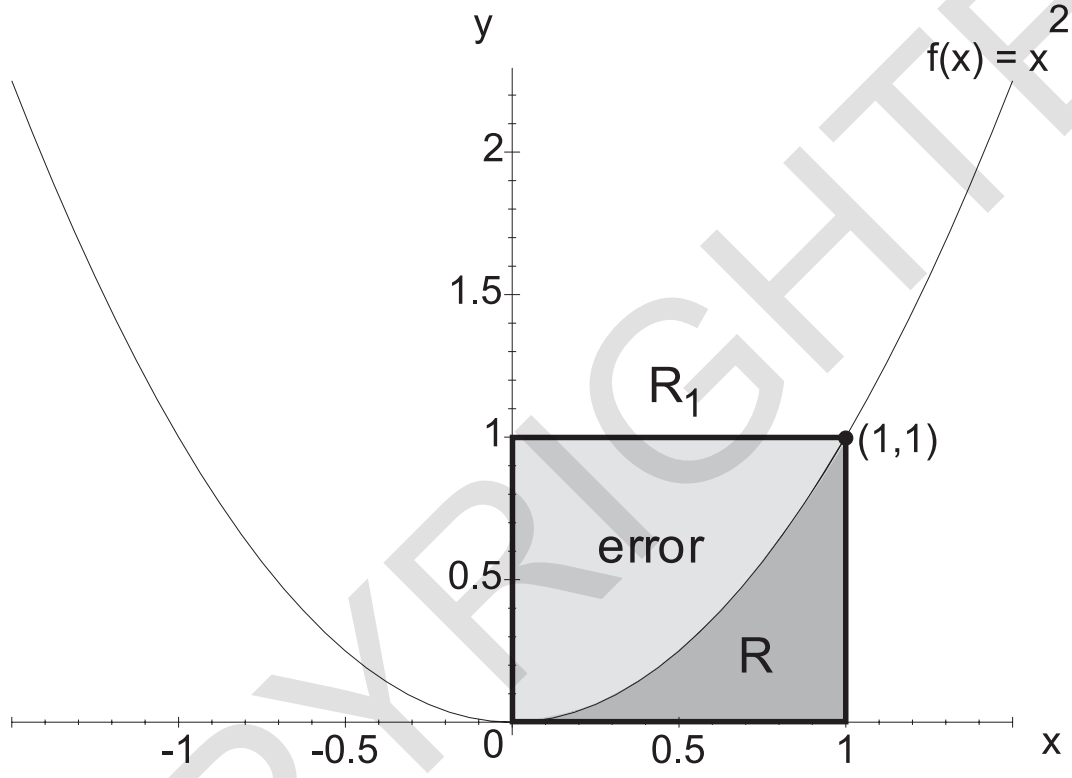
Area Under Curves



Estimate $= R_1$

$$= length \times height = 1 \times f(1) = 1 \times 1^2 = 1$$

Area Under Curves



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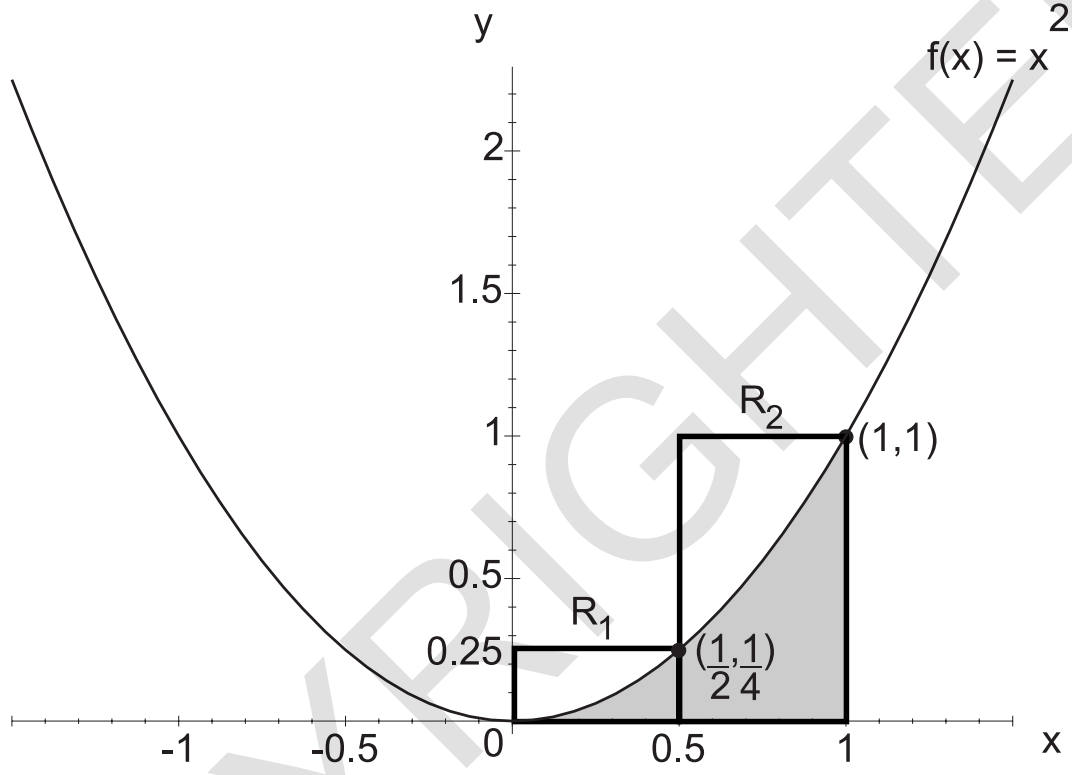
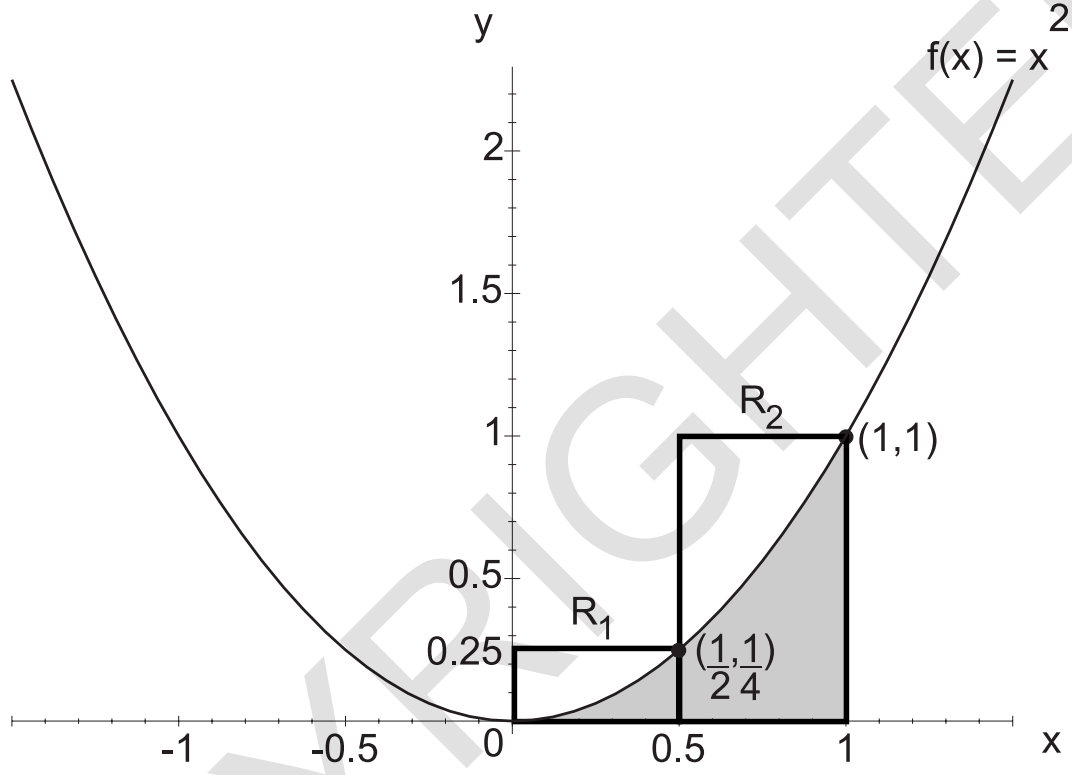
Area Under Curves



Estimate $= \boldsymbol{R_1 + R_2} \;=\; (\frac{\mathbf{1}}{\mathbf{2}} \times \boldsymbol{f}(\frac{\mathbf{1}}{\mathbf{2}})) + (\frac{\mathbf{1}}{\mathbf{2}} \times \boldsymbol{f}(\mathbf{1}))$

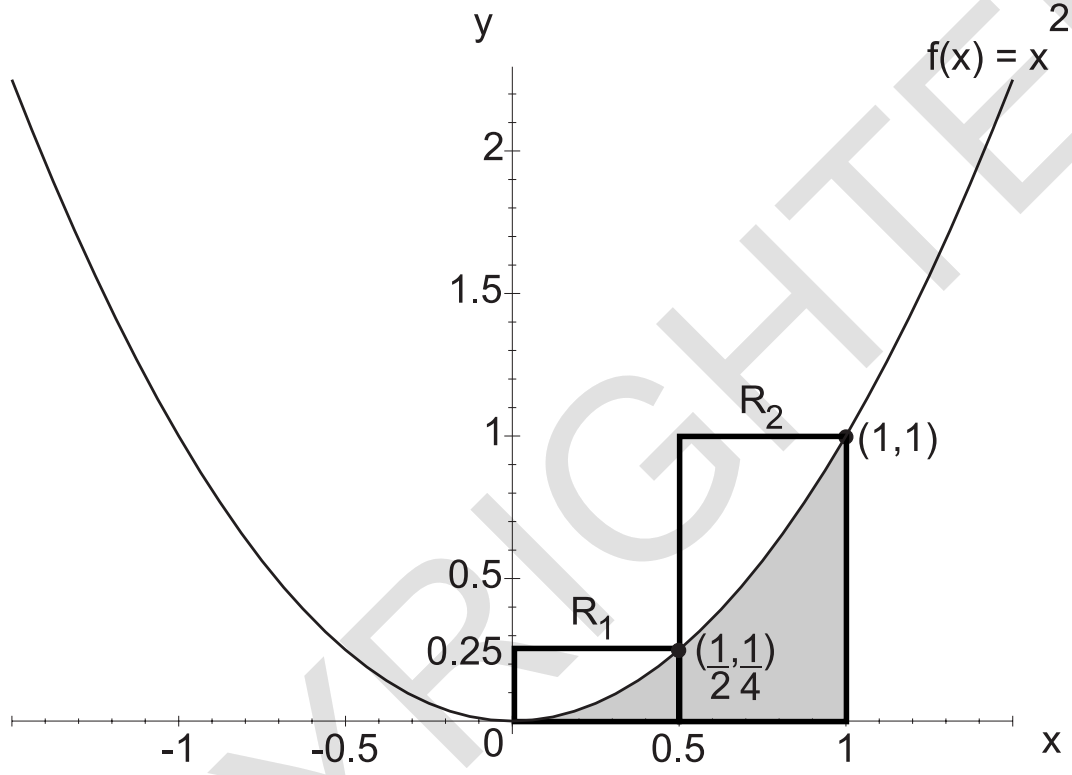
Area Under Curves



Estimate $= R_1 + R_2 = (\frac{1}{2} \times f(\frac{1}{2})) + (\frac{1}{2} \times f(1))$

$= (\frac{1}{2} \times (\frac{1}{2})^2) + (\frac{1}{2} \times 1^2)$

Area Under Curves

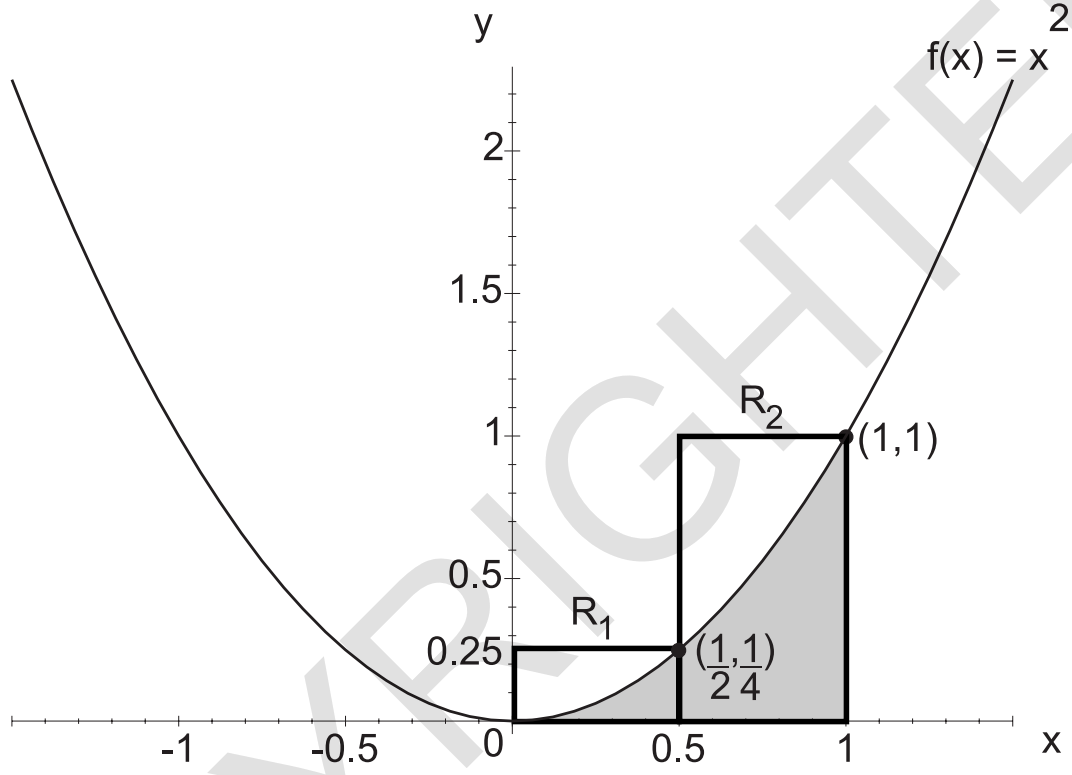


Estimate $= R_1 + R_2$ $= (\frac{1}{2} \times f(\frac{1}{2})) + (\frac{1}{2} \times f(1))$

$= (\frac{1}{2} \times (\frac{1}{2})^2) + (\frac{1}{2} \times 1^2)$

$= (\frac{1}{2} \times \frac{1^2}{2^2}) + (\frac{1}{2} \times \frac{2^2}{2^2})$

Area Under Curves



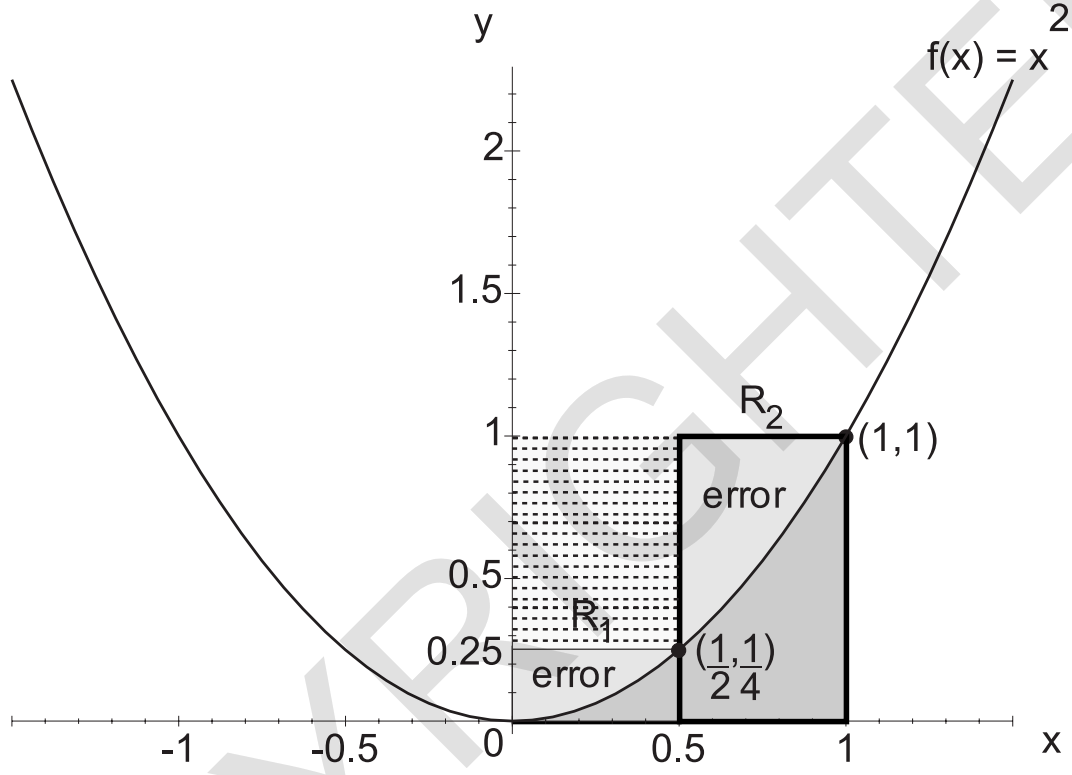
$$\text{Estimate} = R_1 + R_2 = (\frac{1}{2} \times f(\frac{1}{2})) + (\frac{1}{2} \times f(1))$$

$$= (\frac{1}{2} \times (\frac{1}{2})^2) + (\frac{1}{2} \times 1^2)$$

$$= (\frac{1}{2} \times \frac{1^2}{2^2}) + (\frac{1}{2} \times \frac{2^2}{2^2})$$

$$= 0.625$$

Area Under Curves

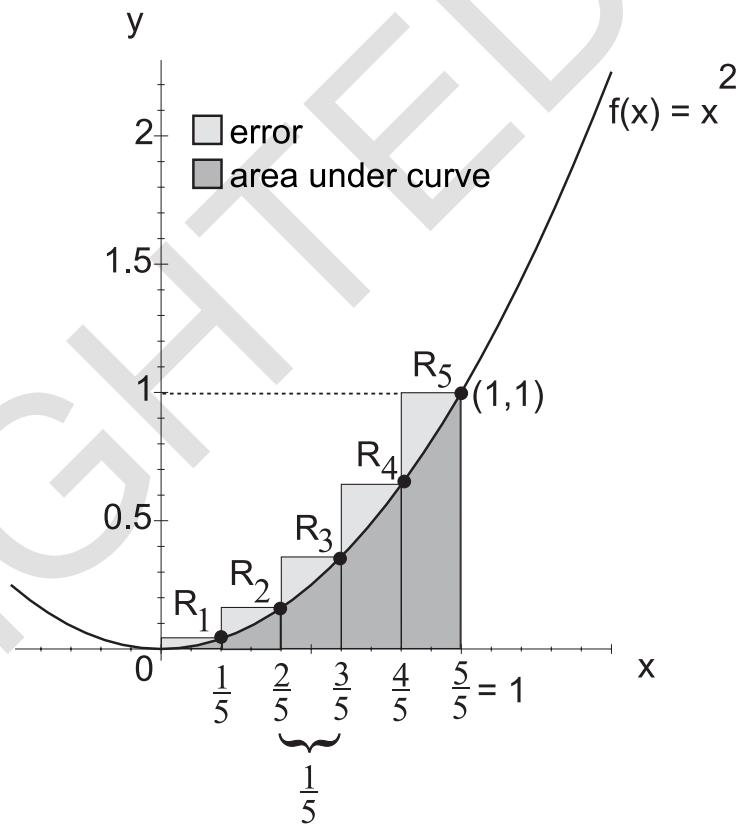


Estimate $= R_1 + R_2$

$$
\begin{aligned}
&= (\frac{1}{2} \times f(\frac{1}{2})) + (\frac{1}{2} \times f(1)) \\
&= (\frac{1}{2} \times (\frac{1}{2})^2) + (\frac{1}{2} \times 1^2) \\
&= (\frac{1}{2} \times \frac{1^2}{2^2}) + (\frac{1}{2} \times \frac{2^2}{2^2}) \\
&= 0.625
\end{aligned}
$$

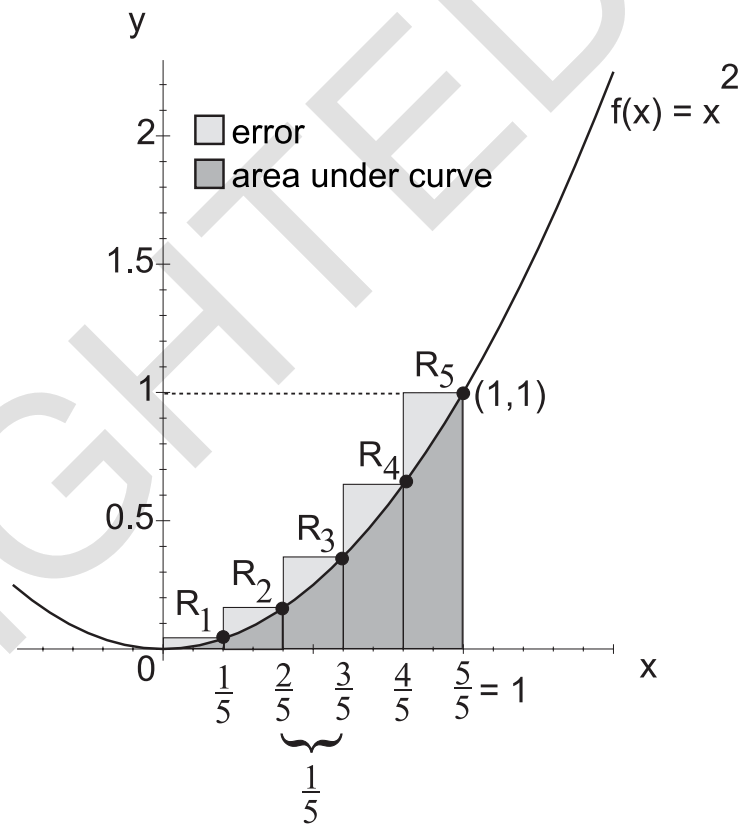
Area Under Curves

Estimate $= R_1 + R_2 + R_3 + R_4 + R_5$



Area Under Curves

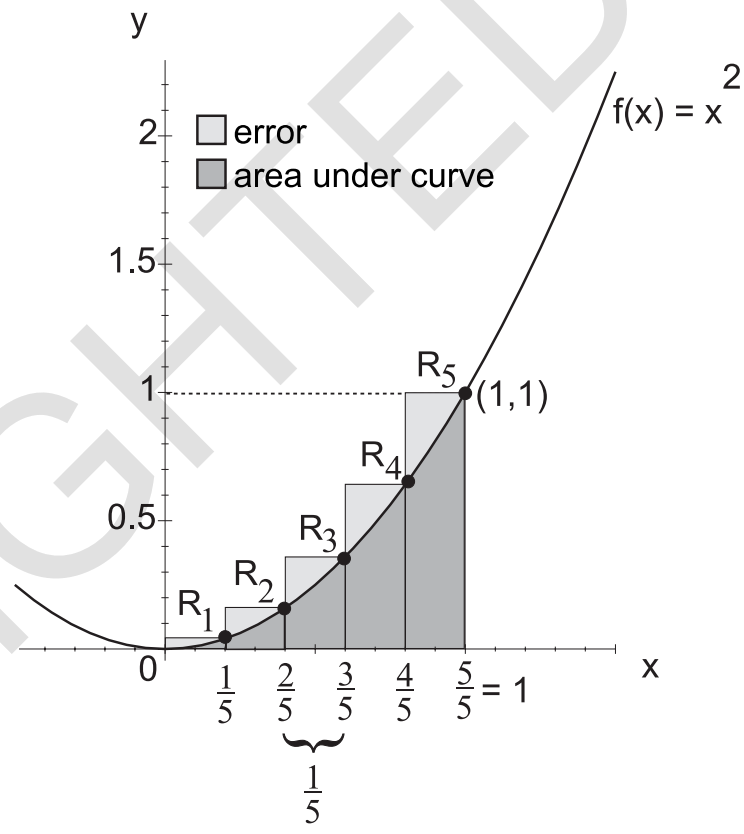
Estimate $= R_1 + R_2 + R_3 + R_4 + R_5$



$$= \quad (\frac{1}{5} \times f(\frac{1}{5})) + (\frac{1}{5} \times f(\frac{2}{5})) + (\frac{1}{5} \times f(\frac{3}{5})) + (\frac{1}{5} \times f(\frac{4}{5})) + (\frac{1}{5} \times f(\frac{5}{5}))$$

Area Under Curves

Estimate $= R_1 + R_2 + R_3 + R_4 + R_5$

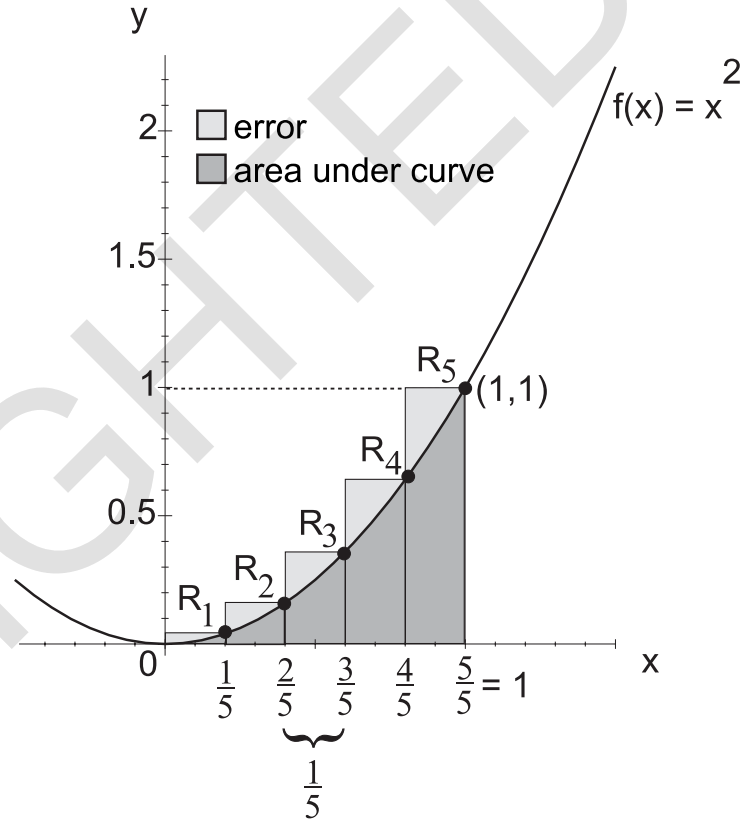


$$= \left(\frac{1}{5} \times f\left(\frac{1}{5}\right)\right) + \left(\frac{1}{5} \times f\left(\frac{2}{5}\right)\right) + \left(\frac{1}{5} \times f\left(\frac{3}{5}\right)\right) + \left(\frac{1}{5} \times f\left(\frac{4}{5}\right)\right) + \left(\frac{1}{5} \times f\left(\frac{5}{5}\right)\right)$$

$$= \left(\frac{1}{5} \times \frac{1^2}{5^2}\right) + \left(\frac{1}{5} \times \frac{2^2}{5^2}\right) + \left(\frac{1}{5} \times \frac{3^2}{5^2}\right) + \left(\frac{1}{5} \times \frac{4^2}{5^2}\right) + \left(\frac{1}{5} \times \frac{5^2}{5^2}\right)$$

Area Under Curves

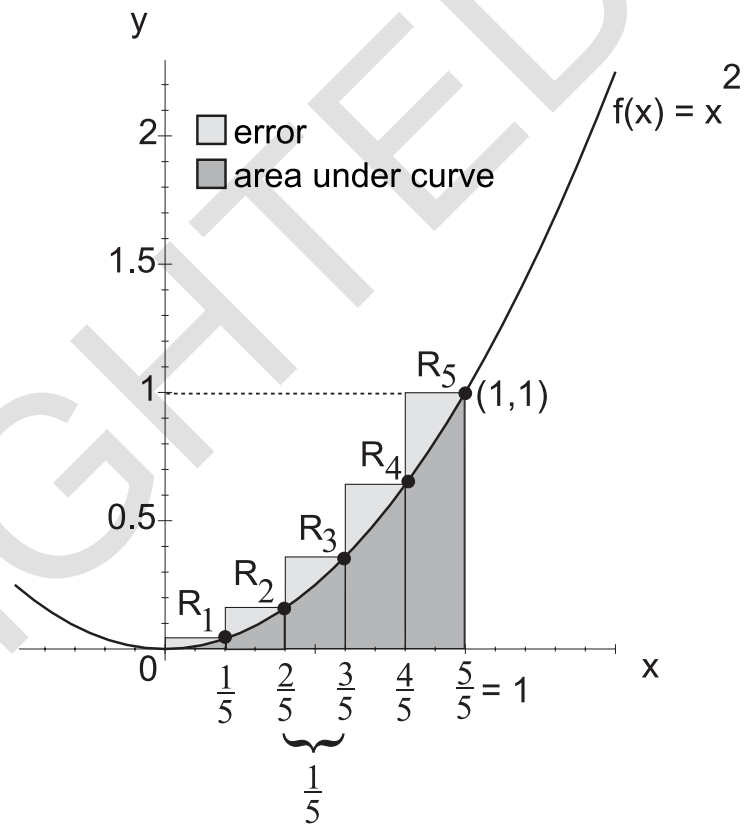
Estimate $= R_1 + R_2 + R_3 + R_4 + R_5$



$$
\begin{aligned}
&= (\frac{1}{5} \times f(\frac{1}{5})) + (\frac{1}{5} \times f(\frac{2}{5})) + (\frac{1}{5} \times f(\frac{3}{5})) + (\frac{1}{5} \times f(\frac{4}{5})) + (\frac{1}{5} \times f(\frac{5}{5})) \\
&= (\frac{1}{5} \times \frac{1^2}{5^2}) + (\frac{1}{5} \times \frac{2^2}{5^2}) + (\frac{1}{5} \times \frac{3^2}{5^2}) + (\frac{1}{5} \times \frac{4^2}{5^2}) + (\frac{1}{5} \times \frac{5^2}{5^2}) \\
&= \frac{1}{5^3} \times (1^2 + 2^2 + 3^2 + 4^2 + 5^2)
\end{aligned}
$$

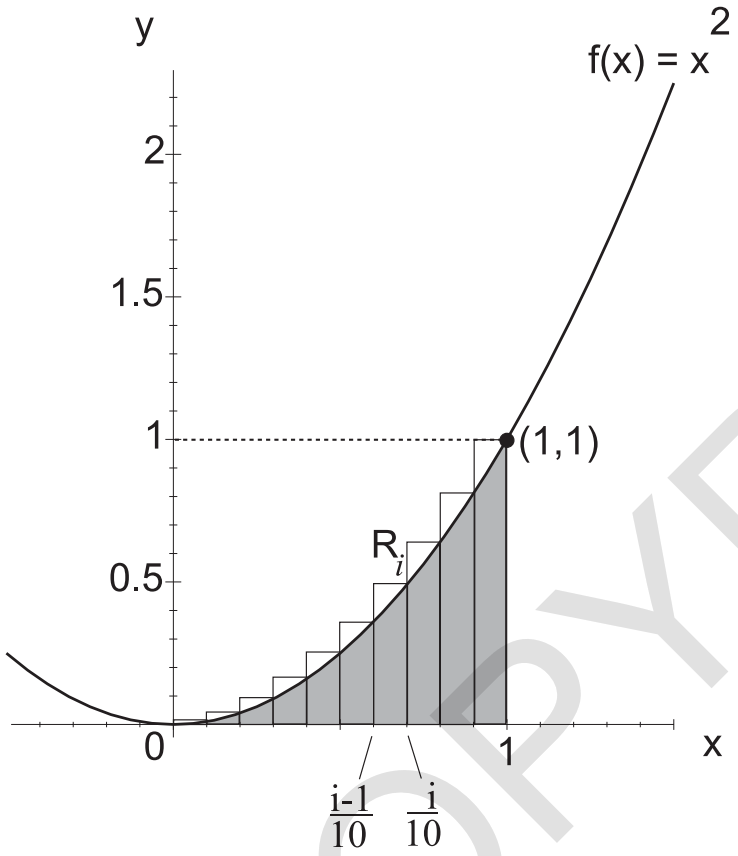
Area Under Curves

Estimate $= R_1 + R_2 + R_3 + R_4 + R_5$



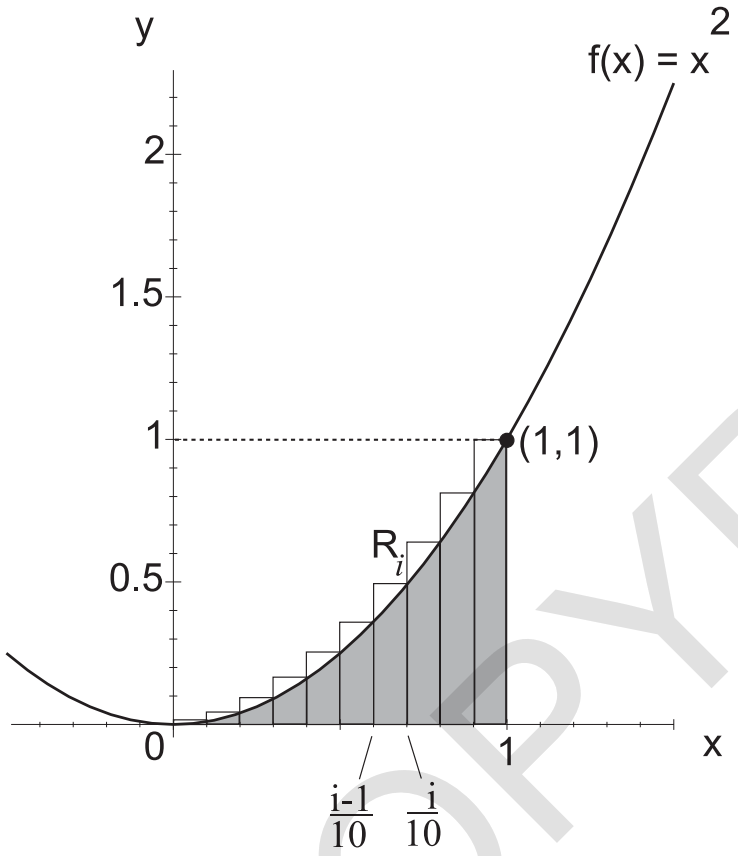
$$
\begin{aligned}
&= (\frac{1}{5} \times f(\frac{1}{5})) + (\frac{1}{5} \times f(\frac{2}{5})) + (\frac{1}{5} \times f(\frac{3}{5})) + (\frac{1}{5} \times f(\frac{4}{5})) + (\frac{1}{5} \times f(\frac{5}{5})) \\
&= (\frac{1}{5} \times \frac{1^2}{5^2}) + (\frac{1}{5} \times \frac{2^2}{5^2}) + (\frac{1}{5} \times \frac{3^2}{5^2}) + (\frac{1}{5} \times \frac{4^2}{5^2}) + (\frac{1}{5} \times \frac{5^2}{5^2}) \\
&= \frac{1}{5^3} \times (1^2 + 2^2 + 3^2 + 4^2 + 5^2) \\
&= 0.44
\end{aligned}
$$

Area Under Curves



$$\sum_{i=1}^{10} R_i = \sum_{i=1}^{10} \frac{1}{10} \times \frac{i^2}{10^2}$$

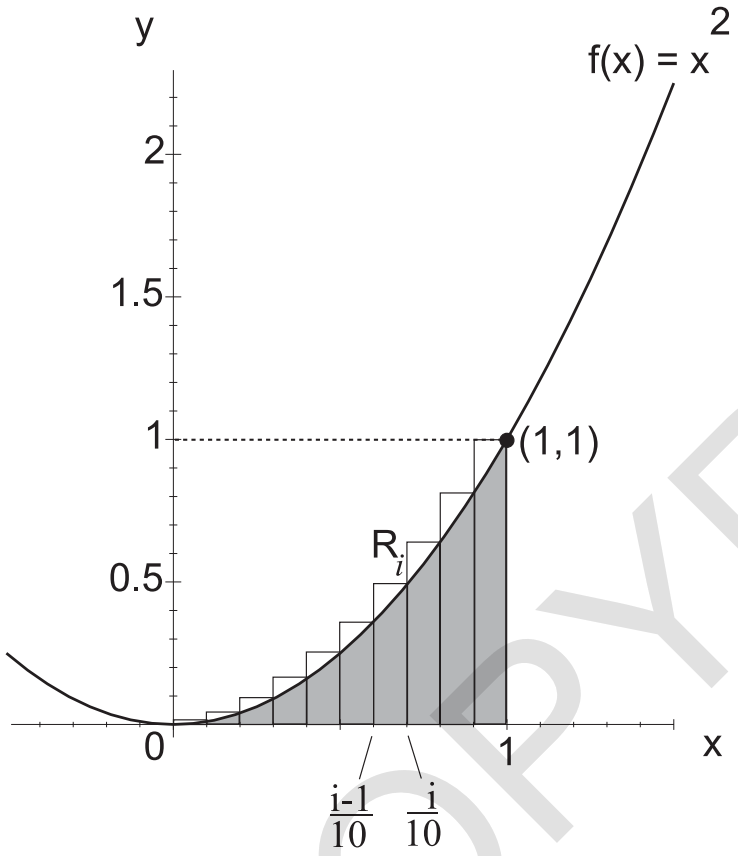
Area Under Curves



$$\sum_{i=1}^{10} R_i = \sum_{i=1}^{10} \frac{1}{10} \times \frac{i^2}{10^2}$$

$$= \frac{1}{10^3} \times \sum_{i=1}^{10} i^2$$

Area Under Curves

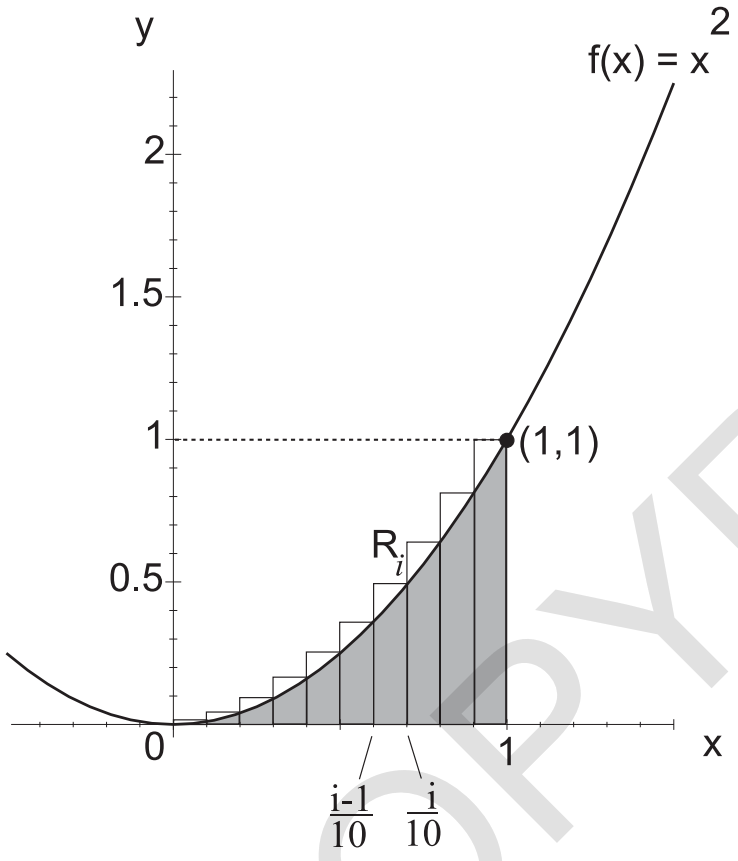


$$\sum_{i=1}^{10} R_i = \sum_{i=1}^{10} \frac{1}{10} \times \frac{i^2}{10^2}$$

$$= \frac{1}{10^3} \times \sum_{i=1}^{10} i^2$$

$$= \frac{1}{10^3} \times \frac{(10)(10+1)(2(10)+1)}{6}$$

Area Under Curves



$$\sum_{i=1}^{10} R_i = \sum_{i=1}^{10} \frac{1}{10} \times \frac{i^2}{10^2}$$

$$= \frac{1}{10^3} \times \sum_{i=1}^{10} i^2$$

$$= \frac{1}{10^3} \times \frac{(10)(10+1)(2(10)+1)}{6}$$

$$= 0.385$$

Area Under Curves

With 1000 subintervals the area estimate is:

$$\sum_{i=1}^{1000} R_i = \sum_{i=1}^{1000} \frac{1}{1000} \times \frac{i^2}{1000^2}$$

Area Under Curves

With 1000 subintervals the area estimate is:

$$\begin{aligned}\sum_{i=1}^{1000} R_i &= \sum_{i=1}^{1000} \frac{1}{1000} \times \frac{i^2}{1000^2} \\ &= \frac{1}{1000^3} \times \sum_{i=1}^{1000} i^2\end{aligned}$$

Area Under Curves

With 1000 subintervals the area estimate is:

$$\begin{aligned}
\sum_{i=1}^{1000} R_i &= \sum_{i=1}^{1000} \frac{1}{1000} \times \frac{i^2}{1000^2} \\
&= \frac{1}{1000^3} \times \sum_{i=1}^{1000} i^2 \\
&= \frac{1}{1000^3} \times \frac{(1000)(1000+1)(2(1000)+1)}{6}
\end{aligned}$$

Area Under Curves

With 1000 subintervals the area estimate is:

$$\begin{aligned}
\sum_{i=1}^{1000} R_i &= \sum_{i=1}^{1000} \frac{1}{1000} \times \frac{i^2}{1000^2} \\
&= \frac{1}{1000^3} \times \sum_{i=1}^{1000} i^2 \\
&= \frac{1}{1000^3} \times \frac{(1000)(1000+1)(2(1000)+1)}{6} \\
&= 0.3338335
\end{aligned}$$

Area Under Curves

With n subdivisions the area estimate is:

$$\begin{aligned}
\sum_{i=1}^{n} R_i &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3}\frac{(n)(n+1)(2(n)+1)}{6} \\
&= \frac{\frac{1}{n^3}(2n^3+3n^2+n)}{6} \\
&= \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}
\end{aligned}$$

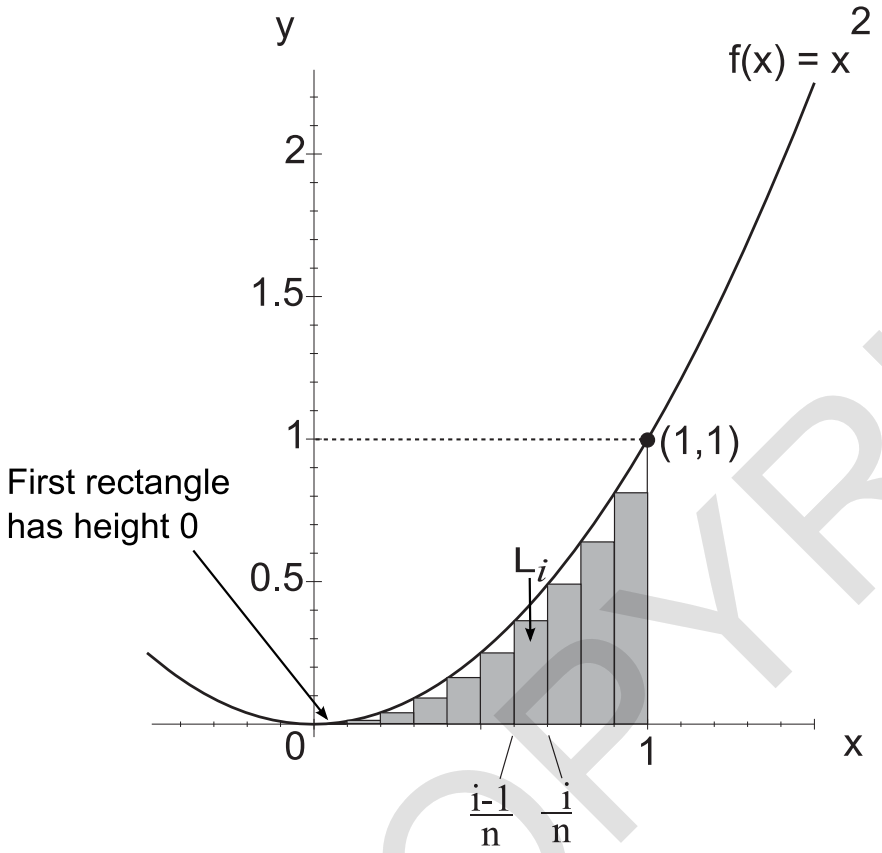
Key Observation: We have

$$\lim_{n\to\infty}\sum_{i=1}^{n} R_i = \lim_{n\to\infty}\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6} = \frac{1}{3}$$

Area Under Curves

Number of Subintervals (Rectangles)	Length of Subinterval (Width of Rectangle)	Estimate for Area under Curve
1	1	1
2	$\frac{1}{2}$	0.625
5	$\frac{1}{5}$	0.44
10	$\frac{1}{10}$	0.385
1000	$\frac{1}{1000}$	0.3338335
approaches ∞	approaches 0	approaches $\frac{1}{3}$

Area Under Curves



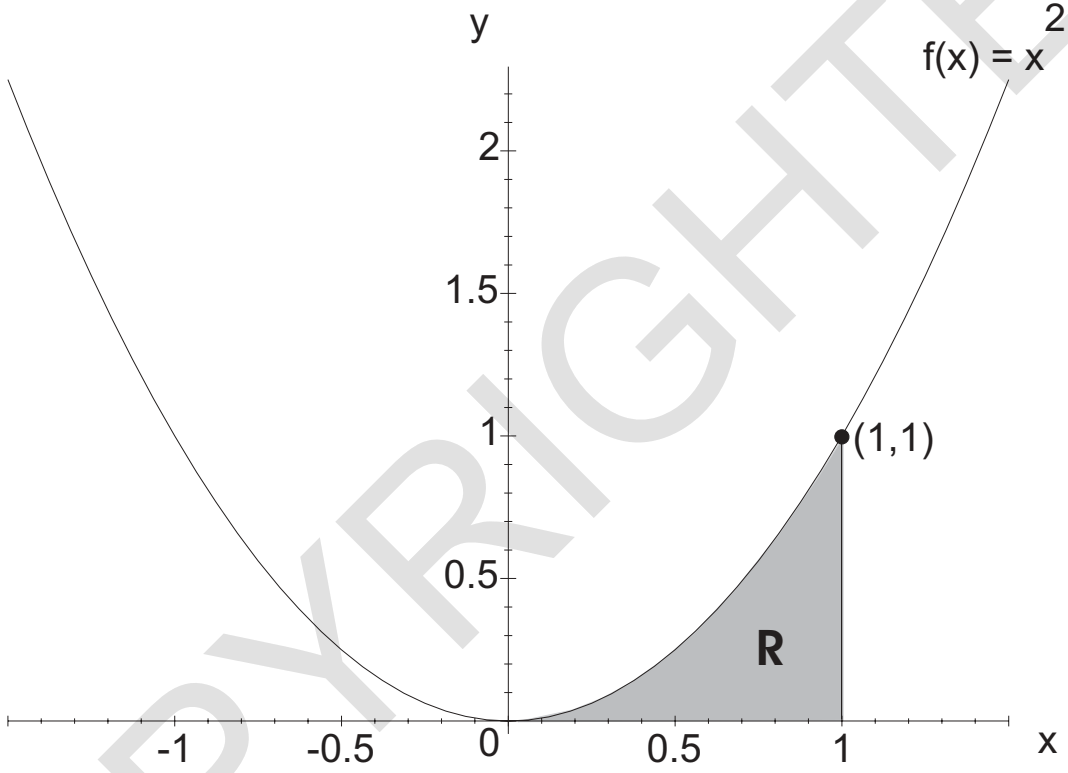
$$\sum_{i=1}^{n} L_i = \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2$$

$$= \frac{1}{n^3} \frac{(n-1)(n+1-1)(2(n-1)+1)}{6}$$

$$= \frac{1}{n^3} \left(\frac{2n^3 - 3n^2 + n}{6} \right)$$

$$= \frac{2 - \frac{3}{n} + \frac{1}{n^2}}{6} \to \frac{1}{3}$$

Area Under Curves



Conclusion:

$$\text{Area} = \frac{1}{3}$$