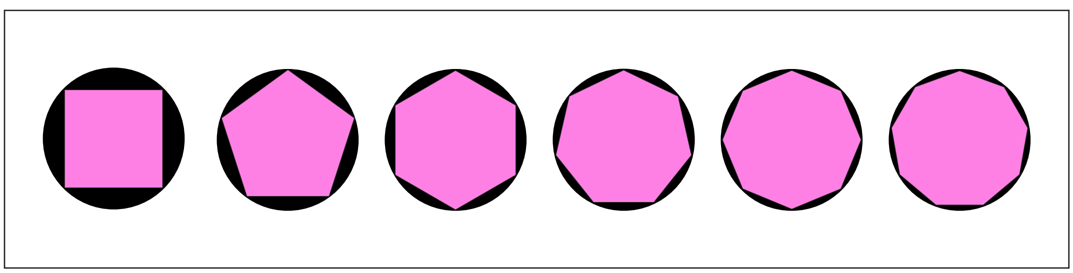
Math 138

Supplementary Course Notes

Curves

Barbara A. Forrest and Brian E. Forrest



September 1, 2019

Author Contact Information:

Barbara Forrest (baforres@uwaterloo.ca)

Brian Forrest (beforres@uwaterloo.ca)

Table of Contents

Chapter 1

Curves

Up until now all of the functions we have dealt with have been real-valued. In this chapter, we will introduce *vector-valued functions*.

1.1 Introduction to Vector-Valued Functions

Vector-valued functions are functions whose domain is contained in $\mathbb{R}$ but whose range is in $\mathbb{R}^n$. To denote these types of functions , we write

$$F(t) : I \subseteq \mathbb{R} \to \mathbb{R}^n$$

where I is some interval in $\mathbb{R}$.

For each such function, there are n real-valued functions $f_1(t), f_2(t), f_3(t), \cdots, f_n(t)$ such that

$$F(t) = (f_1(t), f_2(t), f_3(t), \cdots, f_n(t))$$

The f_i's are called the *coordinate* (or *component*) *functions* for F. We will see that many of the properties of F are inherited from its coordinate functions.

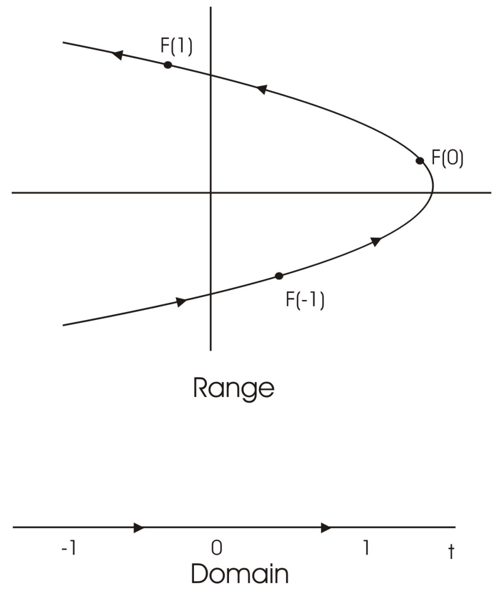
While n could be any positive integer, we will primarily focus on the case $n = 2$. In this case, there are only two coordinate functions that identify the x and y coordinates, respectively, for $F(t)$. For this reason we often write

$$F(t) = (x(t), y(t)).$$

The best way to visualize a vector-valued function is as follows:

Think of t as representing time. Then $F(t)$ can be thought of as *the position of a particle at time t*. In this analogy, the range of $F(t)$ can be viewed as the *path* on which the particle travels.

Unlike real-valued functions, when we study vector-valued functions we will usually focus only on what the range looks like. For this reason, vector-valued functions are often called *curves*.



You will notice that we have included arrows on the diagram. This is often done to provide the curve with an *orientation*. In the analogy of our moving particle, they tell us the direction that the particle is moving in as time moves forward.

In all further examples, the domain will be omitted from the diagram representing the curve.

EXAMPLE 1 Draw the curve $F(t) = (\cos(t), \sin(t))$ with $t \in [0, 2\pi]$.

Drawing curves is somewhat of an art. There is no fixed algorithm to follow as there is for graphing real-valued functions. However, it is often helpful to try and identify the path on which the curve lies. This can be done for $F(t)$ by making the following observation:

The coordinate functions are

$$x(t) = \cos(t)$$

and

$$y(t) = \sin(t).$$

The Pythagorean Theorem shows that

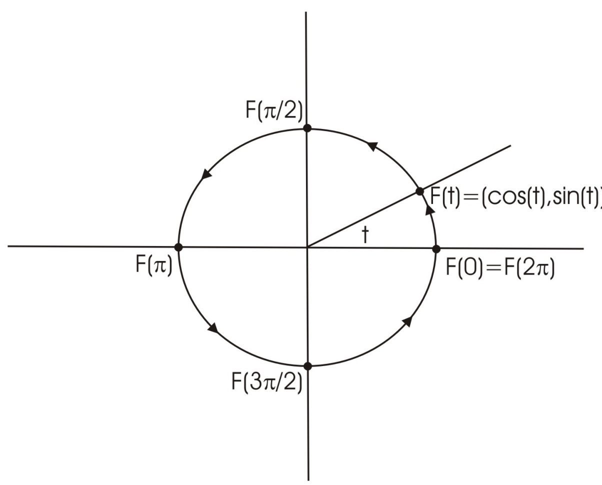
$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1.$$

Geometrically, this means that the points on the curve will lie on the circle with radius 1 centered at $(0, 0)$. Moreover, if we think of t as an angle measured counter-clockwise from the positive x-axis, then $(\cos(t), \sin(t))$ represents the location on the unit circle associated with a ray at angle t. Once we know this we can plot a few points to get a feel for how the curve appears. We will use the analogy of a particle moving in the plane to help us.

The following are sample values of $F(t)$:

$$\begin{aligned}
t = 0, \qquad F(0) &= (\cos(0), \sin(0)) \\
&= (1, 0) \\[1em]
t = \tfrac{\pi}{2}, \qquad F(\tfrac{\pi}{2}) &= (\cos(\tfrac{\pi}{2}), \sin(\tfrac{\pi}{2})) \\
&= (0, 1) \\[1em]
t = \pi, \qquad F(\pi) &= (\cos(\pi), \sin(\pi)) \\
&= (-1, 0) \\[1em]
t = \tfrac{3\pi}{2}, \qquad F(\tfrac{3\pi}{2}) &= (\cos(\tfrac{3\pi}{2}), \sin(\tfrac{3\pi}{2})) \\
&= (0, -1) \\[1em]
t = 2\pi, \qquad F(2\pi) &= (\cos(2\pi), \sin(2\pi)) \\
&= (1, 0)
\end{aligned}$$

Therefore, as t moves from 0 to $\frac{\pi}{2}$ the particle moves counter-clockwise from it initial position at $F(0) = (1, 0)$ along the arc of the circle to $F(\frac{\pi}{2}) = (0, 1)$. It continues along until it hits $(-1, 0)$ at $t = \pi$, then $(0, -1)$ at $t = \frac{3\pi}{2}$, until finally returning to $(1, 0)$ at $t = 2\pi$.



◀

It would be tempting to identify the curve $F(t) = (\cos(t), \sin(t))$ with the circle. However, we must always keep in mind that what we are actually looking at is a *function*. To see why this is relevant, consider the curve

$$G(t) = (\cos(2t), \sin(2t))$$

defined on the interval $[0, \pi]$.

This time the coordinate functions are

$$x(t) = \cos(2t)$$

and

$$y(t) = \sin(2t).$$

However, we still have that

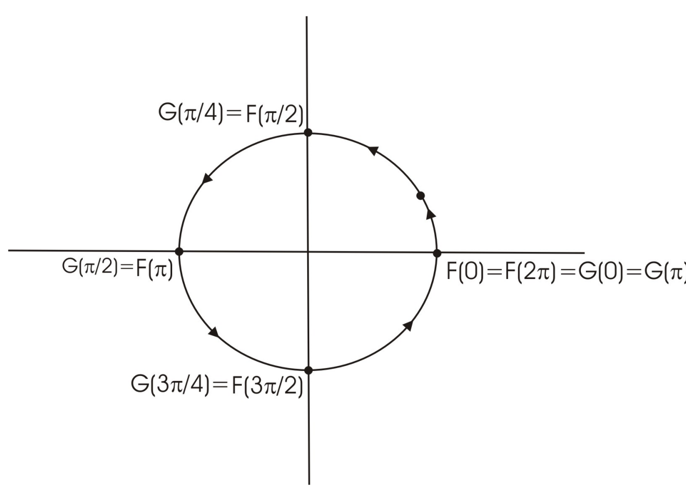
$$x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1.$$

Hence, this curve has a range that also lies on the circle of radius 1 centered at the origin. Moreover, when

$$\begin{array}{llcl}
t = 0, & G(0) & = & (\cos(0), \sin(0)) \\
 & & = & (1, 0) \\
 & & & \\
t = \frac{\pi}{4}, & G(\frac{\pi}{4}) & = & (\cos(2\frac{\pi}{4}), \sin(2\frac{\pi}{4})) \\
 & & = & (0, 1) \\
 & & & \\
t = \frac{\pi}{2}, & G(\frac{\pi}{2}) & = & (\cos(2\frac{\pi}{2}), \sin(2\frac{\pi}{2})) \\
 & & = & (-1, 0) \\
 & & & \\
t = \frac{3\pi}{4}, & G(\frac{3\pi}{4}) & = & (\cos(2\frac{3\pi}{4}), \sin(2\frac{3\pi}{4})) \\
 & & = & (0, -1) \\
 & & & \\
t = \pi, & G(\pi) & = & (\cos(2\pi), \sin(2\pi)) \\
 & & = & (1, 0)
\end{array}$$

Therefore, as t moves from 0 to $\frac{\pi}{4}$ the particle moves counter-clockwise from its initial position at $(1, 0)$ along the arc of the circle to $G(\frac{\pi}{4}) = (0, 1)$, then on to $(-1, 0)$ at $t = \frac{\pi}{2}$, $(0, -1)$ at $t = \frac{3\pi}{4}$ and finally back to $(1, 0)$ at $t = \pi$.

If we sketch the range of both $F(t)$ and $G(t)$, in both cases we get the circle of radius 1.



However, as functions they are different because they assign different values to the same point. In fact, their domains are also different.

Furthermore, our particle analogy can help us identify a fundamental difference in these two functions that we will focus on later. In the first case, with $F(t)$ the particle makes one full revolution of the circle in 2π units of time. In the second case, with $G(t)$, the particle makes the full revolution in half the time. Therefore, we would expect that a particle with position governed by $G(t)$ would be moving at twice the speed as a particle governed by $F(t)$. In this sense the motion of the two particles is quite different, despite the fact that they have traveled on the exact same path.

This brief discussion suggests that the *velocity* of a particle is somehow built into the position function. Since we generally think of velocity as the *derivative of position,* this leads us to speculate that there should be a notion of differentiation for vector-valued functions. In fact, there is and it is particularly easy to realize. However, before we can do so, we need a short discussion on limits for vector-valued functions which we will discuss after some additional examples of curves.

EXAMPLE 2 Let $f : I \to\to \mathbb{R}$. We can build a curve from f as follows:

Let

$$F(t) = (t, f(t))$$

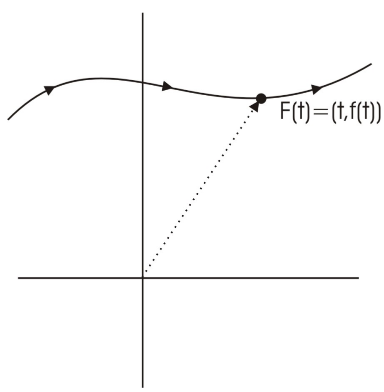
Then $F(t)$ is a curve with coordinate functions

$$x(t) = t$$

and

$$y(t) = f(t).$$

The range of this curve is simply the graph of $f(x)$.



◀

This example shows us that every real-valued function can be associated in a natural way with a curve. However, we will see later that there are some important differences when viewing the function $f(t)$ as a curve, rather than as a function of one variable.

EXAMPLE 3 Sketch the curve $F(t) = (t^3, t^2)$.

Once again, we will begin with the coordinate functions. The coordinate functions are

$$x = x(t) = t^3$$

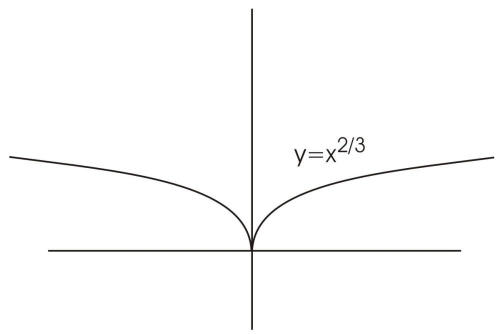
and

$$y = y(t) = t^2.$$

If we take the expression $x = t^3$ and solve for t this gives us $t = x^{\frac{1}{3}}$. We then substitute for t in the second expression to get

$$y = t^2 = (x^{\frac{1}{3}})^2 = x^{\frac{2}{3}}$$

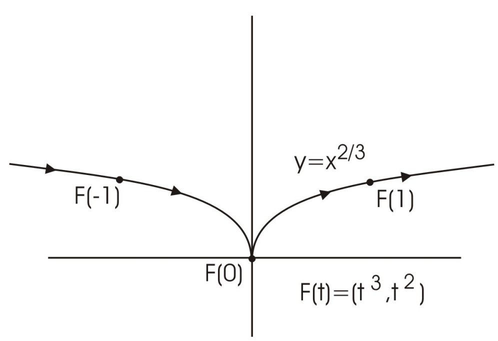
This means that the curve sits on the graph of the function $y = x^{\frac{2}{3}}$. This information helps a great deal since we have lots of tools for graphing functions. Indeed, the graph of $y = x^{\frac{2}{3}}$ looks as follows:



Since the range of $x(t) = t^3$ is all of $\mathbb{R}$, we can achieve every possible x value in the domain of $y = x^{\frac{2}{3}}$, so the curve will actually pass through every point on the graph. Moreover, since $x(t) = t^3$ is $1-1$, we pass through each point only once.

As t goes from $-\infty$ to 0, the x-coordinate is negative and increasing from $-\infty$ to 0 (decreasing in magnitude). From 0 to ∞, the x-coordinate is positive and increasing (increasing in magnitude). Therefore, a particle governed by this curve would move from left to right along the graph of the function $y = x^{\frac{2}{3}}$.

Checking a few points gives $F(-1) = ((-1)^3, (-1)^2) = (-1, 1)$, $F(0) = ((0)^3, (0)^2) = (0, 0)$, and $F(1) = ((1)^3, (1)^2) = (1, 1)$. Hence, the curve looks like



◀

EXAMPLE 4 Sketch the curve $F(t) = (t^2, t^3 - t)$.

This is a rather complicated curve. There are some immediate observations that we can make. Since $x(t) = t^2 \geq 0$, the range of the curve sits to the right of the y-axis. Moreover, since

$$y(t) = t^3 - t = t(t^2 - 1) = t(t-1)(t+1)$$

the curve will pass through the x-axis three times, at $t = 0, \pm 1$. In these cases, we have $F(-1) = (1, 0)$, $F(0) = (0, 0)$ and $F(1) = (1, 0)$, so the curve passes through $(1, 0)$ twice. By looking at the sign of $y(t)$ we see that the curve will lie below the x-axis when $t \in (-\infty, -1)$, lie above the x-axis on $(-1, 0)$, lie below the x-axis again when $t \in (0, 1)$, and finally lie above the x-axis when $t > 1$.

To get a better sense of how the curve appears, we will again try to find a function that can represent the curve. We have

$$x = x(t) = t^2$$

which we could try to solve for t. In this case, there are two possible solutions. Either

$$t = \sqrt{x}$$

or

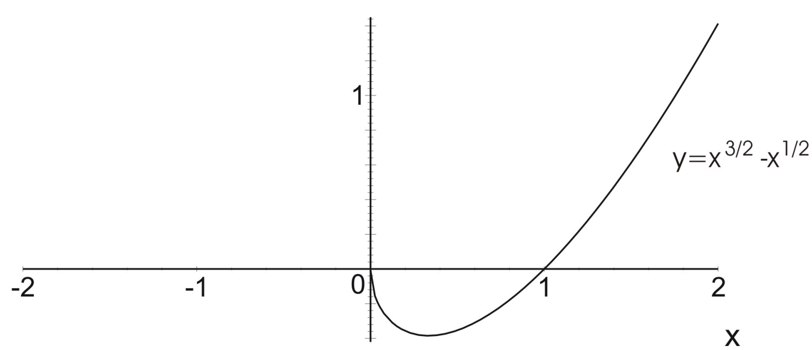
$$t = -\sqrt{x}.$$

It turns out that both are valid. We have that $t = \sqrt{x}$ represents the portion of the curve when $t \geq 0$ and $t = -\sqrt{x}$ gives us the portion of the curve when $t < 0$.

To clarify this statement, first substitute $t = \sqrt{x}$ into the expression for the y-coordinate function. This gives us

$$y = t^3 - t = (\sqrt{x})^3 - \sqrt{x} = x^{\frac{3}{2}} - \sqrt{x}$$

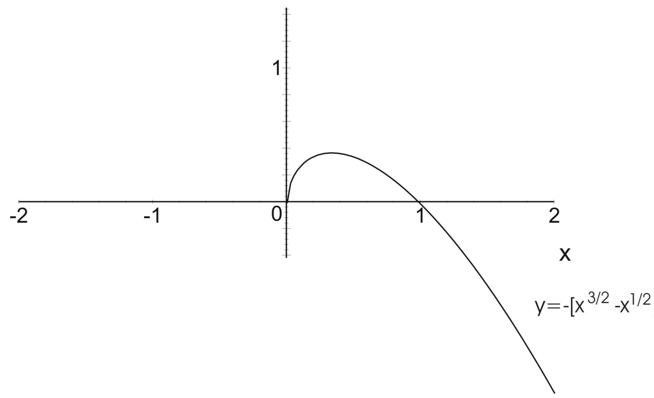
This means that when $t \geq 0$, the curve sits on the graph of this function. The graph looks as follows:



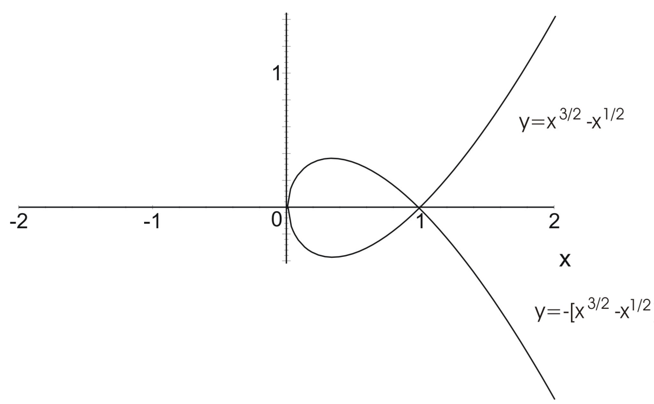
If $t = -\sqrt{x}$ is substituted into the expression for y, we get

$$y = t^3 - t = (-\sqrt{x})^3 - (-\sqrt{x}) = -[x^{\frac{3}{2}} - \sqrt{x}]$$

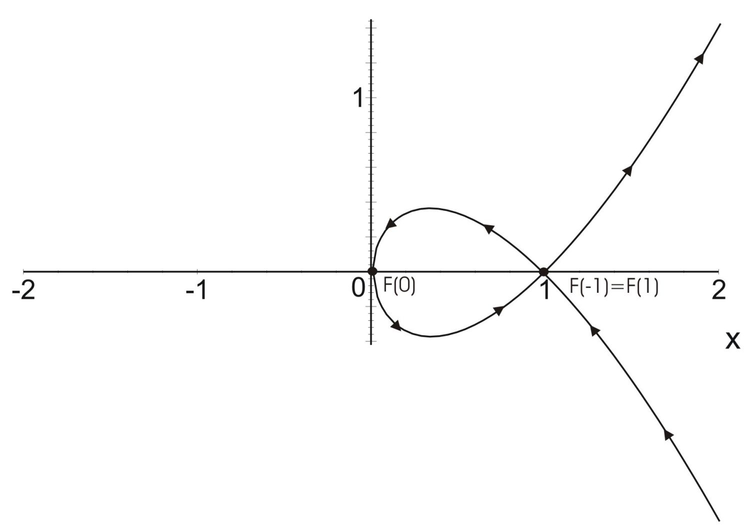
It follows that the portion of the curve corresponding to $t < 0$ lies on the graph of the function $y = -[x^{\frac{3}{2}} - \sqrt{x}]$ which is a mirror reflection of the graph above.



If we put both of these parts together, we get the path on which the curve will lie.



Finally, to obtain the sketch of the curve, we note that as t goes from $-\infty$ towards 0, the curve comes from the bottom right, crosses the x-axis at $t = -1$ through $(1, 0)$, then loops back through the origin at $t = 0$, returns to $(1, 0)$ when $t = 1$, and then proceeds off towards the top right as t goes to ∞.



1.2 Limits and Continuity for Vector-Valued Functions

Formally, we can define the notion of a limit for a function $F : \mathbb{R} \to \mathbb{R}^2$ as follows:

DEFINITION

Limit of a Vector-Valued Function

We say that $L = (L_1, L_2)$ is the *limit of the vector-valued function $F(t)$ as t approaches t_0* if for every positive tolerance $\epsilon > 0$ there is a cutoff distance $\delta > 0$ such that if the distance from t to t_0 is less than δ and $t \neq t_0$, then $F(t)$ approximates L with an error that is less than ϵ. That is, the distance from $F(t)$ to L is less than ϵ.

In this case, we write

$$\lim_{t \to t_0} F(t) = L.$$

It is not the purpose of this section to revisit in detail the formal definition of limit for vector-valued functions. Rather, we make the following observation. If $F(t) = (x(t), y(t))$, then $F(t)$ will be *very close* to $L = (L_1, L_2)$ if and only if

1. $x(t)$ is *very close* to L_1.
2. $y(t)$ is *very close* to L_2.

This provides us with the intuitive justification for the following theorem which will be the key to everything in this section.

THEOREM 1

Limit Theorem for Vector-Valued Functions

Let $F(t) = (x(t), y(t))$ and $L = (L_1, L_2)$. Then

$$\lim_{t \to t_0} F(t) = L$$

if and only if

$$\lim_{t \to t_0} x(t) = L_1$$

and

$$\lim_{t \to t_0} y(t) = L_2.$$

Note: The previous theorem tells us that we can evaluate limits for vector-valued functions by looking at each of the component functions one at a time. The theorem is actually quite easy to prove, though we will not do so. Moreover, it also holds for functions with values in $\mathbb{R}^n$. It is extremely useful since we can now apply everything we know about limits in the real-valued case to vector-valued functions. In particular, *all of the normal arithmetic properties for limits will hold.*

We can also define one-sided limits for vector-valued functions in the obvious way. The natural analog of the previous theorem also holds for one-sided limits.

EXAMPLE 5 Let $F(t) = (t-1, \frac{\sin(t)}{t})$. Find $\lim\limits_{t\to 0} F(t)$.

We have that

$$x(t) = t - 1$$

and

$$y(t) = \frac{\sin(t)}{t}.$$

It is clear that

$$\lim_{t\to 0} x(t) = \lim_{t\to 0} t - 1 = -1.$$

The Fundamental Trigonometric Limit shows that

$$\lim_{t\to 0} y(t) = \lim_{t\to 0} \frac{\sin(t)}{t} = 1.$$

It follows that

$$\lim_{t\to 0} F(t) = (-1, 1).$$

◀

Just as we can define limits for vector-valued functions, we can also define continuity.

DEFINITION **Continuity of a Vector-Valued Function**

A vector-valued function $F(t)$ is continuous at $t = t_0$ if

1. $\lim\limits_{t\to t_0} F(t)$ exists
2. $\lim\limits_{t\to t_0} F(t_0) = F(t_0)$

Not surprisingly, we have the following theorem that tells us that we need only check the continuity of the component functions to see that $F(t)$ is continuous.

THEOREM 2 **Continuity Theorem for Vector-Valued Functions**

Let $F(t) = (x(t), y(t))$. Then $F(t)$ is continuous at $t = t_0$ if and only if both $x(t)$ and $y(t)$ are continuous at $t = t_0$.

1.3 Derivatives of Vector-Valued Functions: Velocity

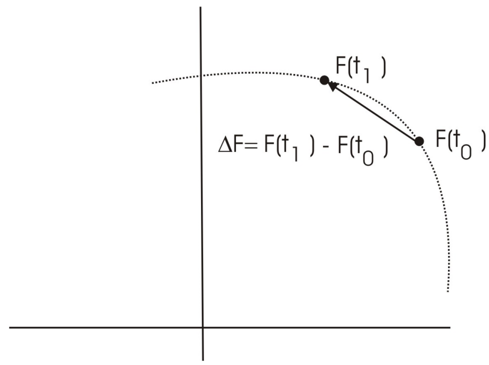
We will return to our analogy of the vector-valued function $F(t)$ representing the position of particle at time t. Suppose now that we wanted to know the *velocity* of the particle. We could proceed as follows:

We know that $F(t)$ represents the position of the particle. Generally speaking, the *average velocity* of the particle between time t_0 and t_1 should be

$$v_{ave} = \frac{\textit{change in position}}{\textit{change in time}}$$

The change in position is the vector

$$\Delta F = F(t_1) - F(t_0)$$



while the change in time is

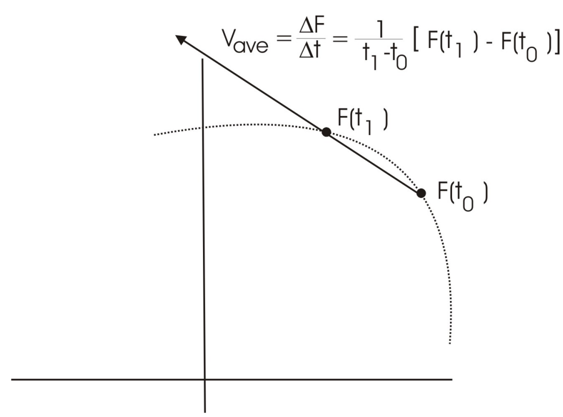
$$\Delta t = t_1 - t_0.$$

Hence

$$v_{ave} = \frac{\Delta F}{\Delta t}$$

However, since ΔF is a vector and Δt is a scalar, what we really mean by this notation is that

$$v_{ave} = \frac{\Delta F}{\Delta t} = \frac{1}{\Delta t}[F(t_1) - F(t_0)].$$



You will notice that v_{ave} is a vector that is parallel to ΔF. It is in the same direction if $\Delta t > 0$ and is in the opposite direction if $\Delta t < 0$. Moreover, since $t_1 = t_0 + \Delta t$, the formula for average velocity is also given by

$$v_{ave} = \frac{1}{\Delta t}[F(t_0 + \Delta t) - F(t_0)].$$

We can now define what we mean by instantaneous velocity.

DEFINITION

Instantaneous Velocity of a Particle

Let $F(t)$ represent the position of a particle at time t. Then the *instantaneous velocity* of the particle at time t_0 is

$$v = \lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[F(t_0 + \Delta t) - F(t_0)].$$

For a closer look at what this means, assume that $F(t) = (x(t), y(t))$. Then

$$\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[F(t_0 + \Delta t) - F(t_0)] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[(x(t_0 + \Delta t), y(t_0 + \Delta t)) - (x(t_0), y(t_0))] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}(x(t_0 + \Delta t) - x(t_0), y(t_0 + \Delta t) - y(t_0)) \\
&= \lim_{\Delta t \to 0} (\frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}, \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t}) \\
&= (\lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}, \lim_{\Delta t \to 0} \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t}) \\
&= (x'(t_0), y'(t_0))
\end{aligned}$$

This shows that the **velocity vector has components that are just the usual derivatives of the component functions of $F(t)$.**

EXAMPLE 6 Suppose that the position of two particles are given by

$$F(t) = (\cos(t), \sin(t))$$

and

$$G(t) = (\cos(2t), \sin(2t))$$

respectively. We have seen that both particles will be traveling around the unit circle in a counter-clockwise direction.

The first particle will reach the point $(0, 1)$ at time $t = \frac{\pi}{2}$, while the second will reach the same point at time $t = \frac{\pi}{4}$. Suppose that we want to calculate the velocity of each particle the instant they reach $(0, 1)$.

For the first particle, we have

$$x(t) = \cos(t)$$

and

$$y(t) = \sin(t)$$

so that

$$x'(t) = -\sin(t)$$

and

$$y'(t) = \cos(t).$$

When $t = \frac{\pi}{2}$, we get that the velocity vector v is given by

$$v = (x'(\frac{\pi}{2}), y'(\frac{\pi}{2})) = (-\sin(\frac{\pi}{2}), \cos(\frac{\pi}{2})) = (-1, 0).$$

For the second particle,

$$x(t) = \cos(2t)$$

and

$$y(t) = \sin(2t).$$

Hence

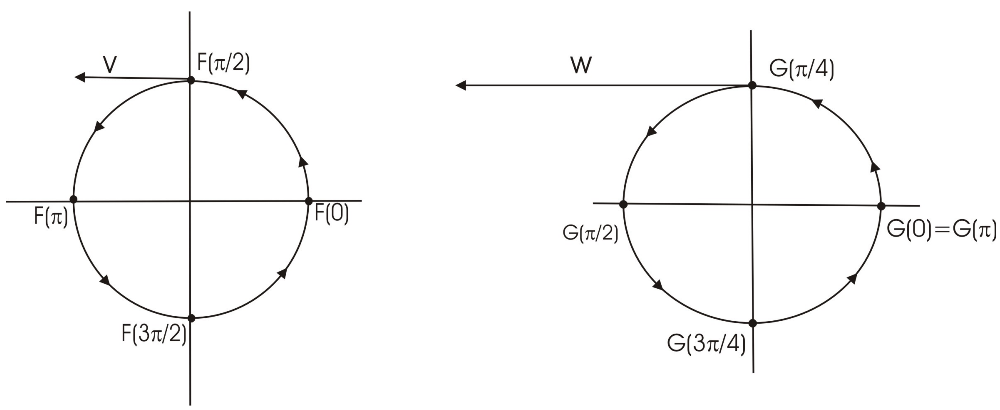
$$x'(t) = -2\sin(2t)$$

and

$$y'(t) = 2\cos(2t).$$

This time we want the velocity vector w at time $t = \frac{\pi}{4}$. Therefore,

$$v = (x'(\frac{\pi}{4}), y'(\frac{\pi}{4})) = (-2\sin(2\frac{\pi}{4}), 2\cos(2\frac{\pi}{4})) = (-2, 0).$$



You will notice that the two vectors are pointing in the same direction and this direction is *tangent* to the circle at the point $(1, 0)$. This is consistent with our physical intuition that tells us that the velocity of a moving object is always tangent to the path of the object.

The second observation is that w is twice as long as v. To see why this is so we note again that the second particle makes one complete revolution of the circle in half the time of the first. Therefore, its *speed* should be twice that of the first. Since *speed is the magnitude of velocity*, the second velocity vector w should have twice the length of the first velocity vector w.

In summary, if the position of a particle is given by $F(t) = (x(t), y(t))$, then the instantaneous velocity at time t_0 is given by $v = (x'(t_0), y'(t_0))$. The vector v points in a direction that is *tangent to the path* of the particle. The length $\|\ v\ \|$ of v is the instantaneous speed of the particle at time t_0. ◀

1.4 Derivatives of Vector-Valued Functions and Tangent Lines

We have just seen how we can define instantaneous velocity for a particle that moves in a manner described by a vector-valued function. Since we know that velocity is in principle the *derivative of position* it would make sense to use what we have learned to formulate a definition for the derivative of a vector-function.

DEFINITION

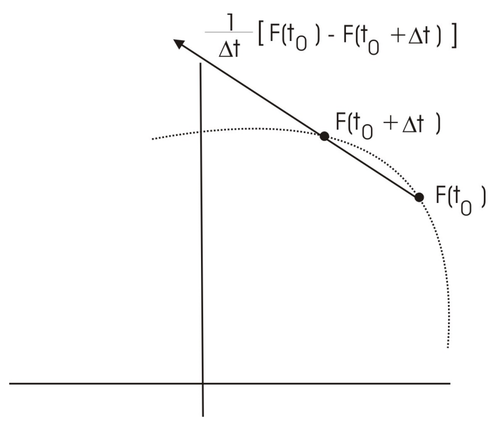
Derivative of a Vector-Valued Function

Given a vector-valued function $F(t)$, we define the derivative of $F(t)$ at t_0 to be

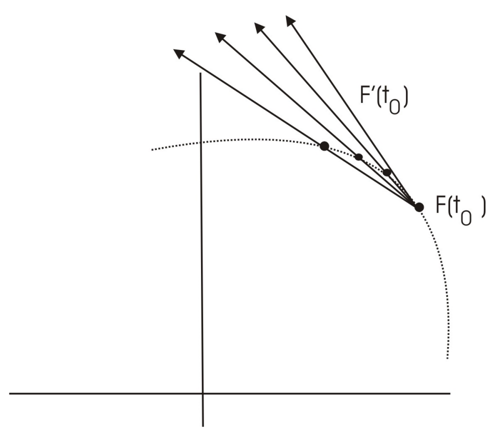
$$F'(t_0) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[F(t_0 + \Delta t) - F(t_0)]$$

provided that this limit exists.

Geometrically, $\frac{1}{\Delta t}[F(t_0 + \Delta t) - F(t_0)]$ represents a **scaled secant vector**.



As such, the derivative $F'(t_0)$ can be viewed as a *limit of scaled secant vectors*. For this reason, we call the derivative vector, *the tangent vector to the curve $F(t)$ at $t = t_0$*.



If the tangent vector $F'(t_0) \neq (0, 0)$, then there is a unique line in the direction of $F'(t_0)$ through the point $F(t_0)$.

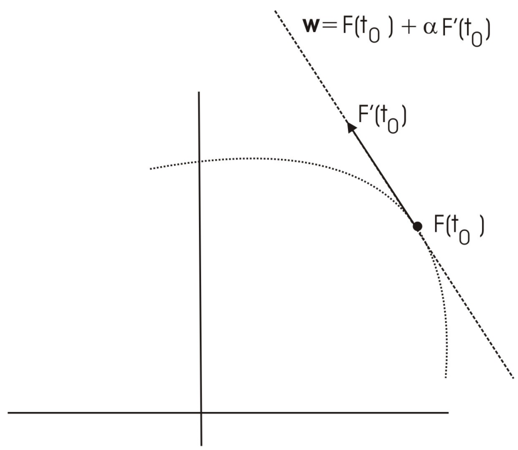
DEFINITION

Tangent Line to a Vector-Valued Function

Assume that $F(t)$ is differentiable at $t = t_0$ and $F'(t_0) \neq (0, 0)$, then the line with vector equation

$$\mathbf{w} = F(t_0) + \alpha F'(t_0)$$

through $F(t_0)$ in the direction of $F'(t_0)$ is called the **tangent line** to the curve $F(t)$ at $t = t_0$.



Tangent Line

We have defined the derivative and used it to define the tangent vector and the tangent line. The next theorem tells us that the derivative of a vector-valued function can be calculated component-wise. The argument that justifies this theorem is exactly the same as we presented in calculating velocities.

THEOREM 3

Differentiation Theorem for Vector-Valued Functions

Let $F(t) = (x(t), y(t))$. Then $F(t)$ is differentiable at $t = t_0$ if and only if both $x(t)$ and $y(t)$ are differentiable at $t = t_0$.

In this case,

$$F'(t_0) = (x'(t_0), y'(t_0))$$

Just as we did for functions of one variable, we can define the *derivative function* by

$$F'(t) = (x'(t), y'(t))$$

for every t at which the derivative exists.

EXAMPLE 7 Find the derivative of $F(t) = (t^3, t^2)$ at $t = 1$. Find the vector equation of the tangent line to the curve at $t = 1$.

We know from the Differentiation Theorem for Vector-valued Functions that

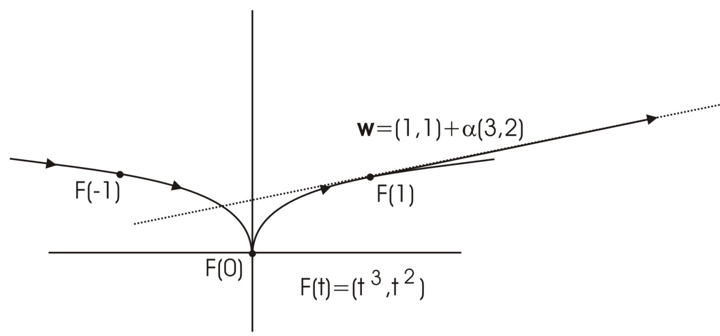
$$F'(t) = (3t^2, 2t).$$

Therefore, $F'(1) = (3, 2)$.

Since $F(1) = (1, 1)$, the vector equation for the tangent line is

$$\mathbf{w} = F(t_0) + \alpha F'(t_0) = (1, 1) + \alpha(3, 2).$$

The following picture shows the curve together with its tangent vector and tangent line at $t = 1$.



◀

We have seen how to find the vector equation of the tangent line. Suppose that we wanted to find the standard equation of this line. Then we would only need to determine the slope since we know the line passes through $F(t_0)$. However, if a line is in the direction of the vector (x, y) and $x \neq 0$, then its slope is

$$m = \frac{y}{x}.$$

If $x = 0$ and $y \neq 0$, the line is vertical.

EXAMPLE 8 Assume that $f(t)$ is a differentiable function and that we define a curve by

$$F(t) = (t, f(t)).$$

Then the function $F(t)$ is differentiable at t with $F'(t) = (1, f'(t))$. The tangent line passes through $F(t) = (t, f(t))$ and has slope

$$m = \frac{f'(t)}{1} = f'(t).$$

This means that for a curve defined from a differentiable function, the tangent line we have just defined agrees exactly with the usual definition of a tangent line.

You will notice that when we defined the tangent line for a vector-valued function, we added in the assumption that $F'(t_0) \neq (0,0)$. This assumption is necessary for good reason:

If $F'(t_0) = (0,0)$, the vector equation of the line would be

$$\mathbf{w} = F(t_0) + \alpha F'(t_0) = F(t_0) + \alpha(0,0) = F(t_0)$$

However, this is a point, **not** a line. ◀

The next example shows that when $F'(t_0) = (0,0)$, a curve can be badly behaved in the sense that despite it being differentiable at t_0, the curve would have no natural tangent line.

EXAMPLE 9 Consider the vector-valued function

$$F(t) = (t^3, t^2).$$

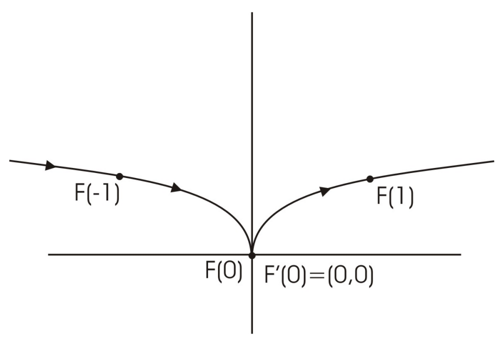
We have seen that

$$F'(t) = (3t^2, 2t).$$

Hence

$$F'(0) = (3(0^2), 2(0)) = (0,0).$$

If we look at the curve, we will see that there is a *sharp point* at the origin.



We would usually expect such a point to **not** be a point of differentiability. However, for curves such as the one above, this may not be the case. For this reason, we will say that a curve is *smooth* at $t = t_0$ if $F(t)$ is differentiable at $t = t_0$ and if $F'(t_0) \neq (0,0)$.

◀

The next example shows one more unusual aspect of differentiable curves.

EXAMPLE 10 Let $F(t) = (t^2, t^3 - t)$. Then we have seen that

$$F(-1) = (1,0) = F(1).$$

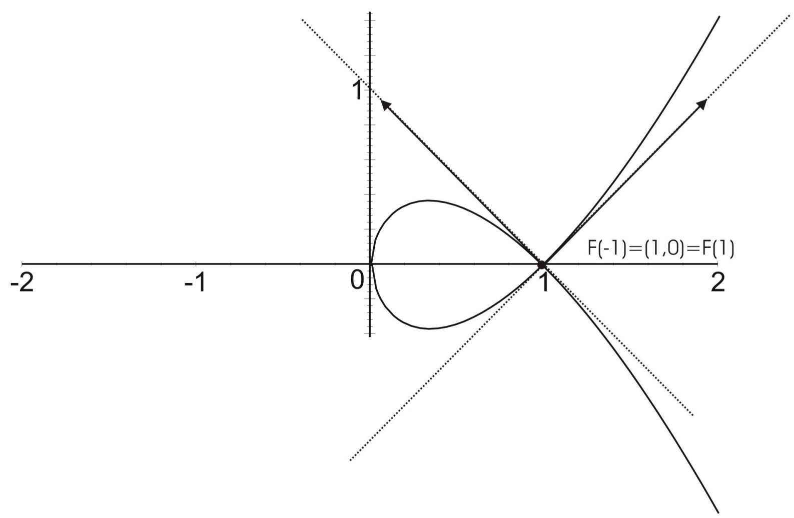
However, since $F'(t) = (2t, 3t^2 - 1)$ we get that

$$F'(-1) = (-2, 2)$$

while

$$F'(1) = (2, 2).$$

Therefore, there are two different tangent vectors at the same location (**though at different values of** t). Similarly, there will be two different tangent lines through $(1, 0)$.



1.5 Linear Approximation for Vector-Valued Functions

Recall that if the real-valued $f(t)$ is differentiable at $t = t_0$, then we can define the linear approximation to $f(t)$ at $t = t_0$ by

$$L_{t_0}(t) = f(t_0) + f'(t_0)(t - t_0).$$

The linear approximation had two very important properties:

1. $L_{t_0}(t_0) = f(t_0)$

   and

2. $L'_{t_0}(t_0) = f'(t_0)$.

Since the linear approximation was such a valuable tool for real-valued functions, we would like to find an analog for vector-valued functions. Indeed, if $F(t)$ is differentiable at t_0, then letting $t = t_0 + \Delta t$ gives us

$$F'(t_0) = \lim_{t \to t_0} \frac{1}{t - t_0}[F(t) - F(t_0)]$$

Therefore if $t \cong t_0$, we have

$$F'(t_0) \cong \frac{1}{t - t_0}[F(t) - F(t_0)]$$

so

$$(t - t_0)F'(t_0) \cong F(t) - F(t_0)$$

and

$$F(t) \cong F(t_0) + (t - t_0)F'(t_0)$$

If we let

$$L_{t_0}(t) = F(t_0) + (t - t_0)F'(t_0),$$

then if $t \cong t_0$,

$$L_{t_0}(t) \cong F(t)$$

Moreover, it is easy to verify that if $F'(t) \neq (0, 0)$, then $L_{t_0}(t)$ has the following properties:

1. $L_{t_0}(t)$ is a vector-valued function with its range being a line.
2. $L_{t_0}(t_0) = F(t_0)$
3. $L'_{t_0}(t) = F'(t_0)$.

Note: The third property requires a little algebra to verify, but it is straight-forward.

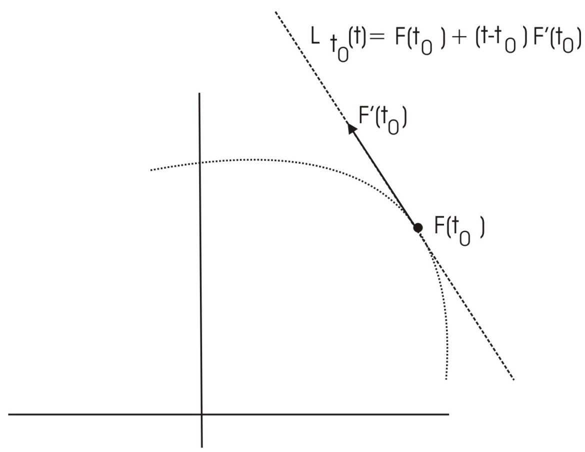
In particular, the range of the vector-valued function $L_{t_0}(t)$ is precisely the tangent line to the curve $F(t)$ at $t = t_0$. This shows that $L_{t_0}(t)$ has all the properties that we are looking for in an analog of the linear approximation. Therefore, the following definition is natural.

DEFINITION

Linear Approximation to a Vector-Valued Function

Let $F(t)$ be a vector-valued function. Assume that $F(t)$ is differentiable at $t = t_0$ and that $F'(t_0) \neq (0, 0)$. Then the *linear approximation to $F(t)$ at $t = t_0$* is the vector-valued function

$$L_{t_0}(t) = F(t_0) + (t - t_0)F'(t_0).$$



Linear Approximation

PROBLEMS:

1. A ball on the end of a one meter long string is swung counter-clockwise in a circle at the rate of 1 revolution every 2 seconds. Assuming that the end of the string is at the origin and that at time $t = 0$ the ball is at the point $(1, 0)$, find a curve $F(t)$ that represents the position of the ball at any given time.

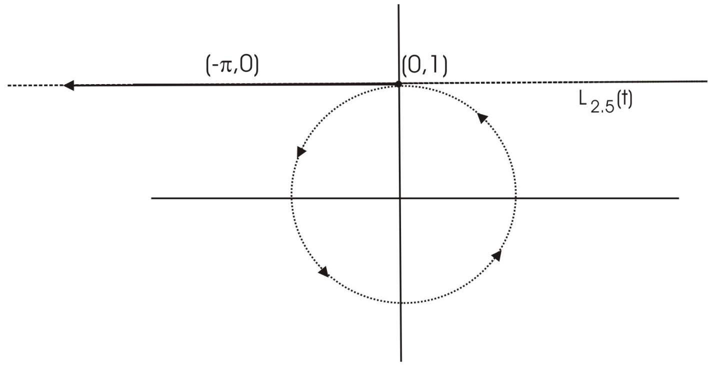
2. Find $F'(2.5)$ and $L_{2.5}(t)$.

3. Assuming that the string is released at $t = 2.5$, find a vector-valued function $G(t)$ that will determine the horizontal position of the ball relative to the origin for $t \geq 2.5$.

SOLUTIONS:

1. First note that the ball will be traveling along a path that is a circle of radius 1. Moreover, the ball is traveling with constant velocity, it begins at $(1, 0)$, and the motion is counter-clockwise. These statements imply that there is a constant k such that
$$F(t) = (\cos(kt), \sin(kt)).$$
However, since the ball makes one revolution every 2 seconds, when $t = 2$ our angle kt must be 2π. Therefore, $k = \pi$ is a solution. Hence, the position of the ball is given by
$$F(t) = (\cos(\pi t), \sin(\pi t)).$$

2. We have
$$F'(t) = (\pi \sin(\pi t), \pi \cos(\pi t)).$$
Therefore
$$\begin{aligned} F'(2.5) &= (-\pi \sin(2.5\pi), \pi \cos(2.5\pi)) \\ &= (-\pi \sin(2\pi + .5\pi), \pi \cos(2\pi + .5\pi)) \\ &= (-\pi \sin(.5\pi), \pi \cos(.5\pi)) \\ &= (-\pi, 0) \end{aligned}$$
We also have that
$$\begin{aligned} F(t) &= (\cos(2.5\pi), \sin(2.5\pi)) \\ &= (\cos(2\pi + .5\pi), \sin(2\pi + .5\pi)) \\ &= (\cos(.5\pi), \sin(.5\pi)) \\ &= (0, 1) \end{aligned}$$
Therefore,
$$\begin{aligned} L_{2.5}(t) &= F(2.5) + (t - 2.5)F'(2.5) \\ &= (0, 1) + (t - 2.5)(-\pi, 0) \end{aligned}$$



3. Once the ball is released it will travel in a straight line that is tangent to its path at the time of release. Moreover, the ball will have constant velocity that is equal to the velocity at the moment it was released. But the velocity at that point was $F'(2.5) = (-\pi, 0)$. This shows that Δt seconds after its release, the ball's position will be
$$G(t) = (0, 1) + \Delta t(-\pi, 0)$$
However, since $\Delta t = t - 2.5$, we get
$$G(t) = (0, 1) + (t - 2.5)(-\pi, 0) = L_{2.5}(t).$$
This shows that the ball's position will be determined by the linear approximation.

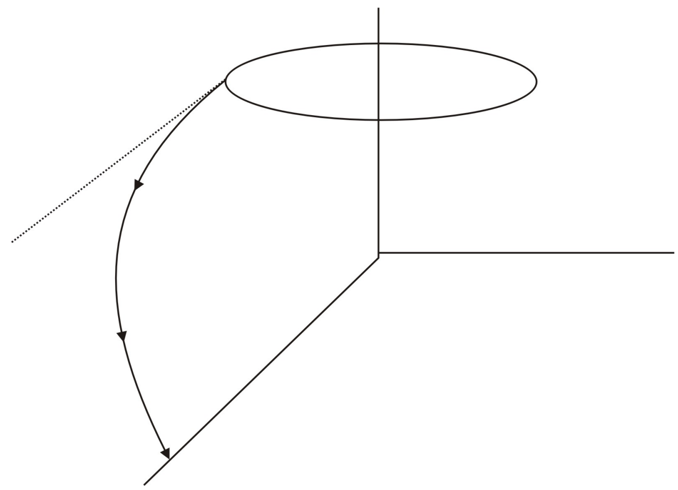
So far everything we have done has been for functions into $\mathbb{R}^2$. However, the next example adds one more dimension to our previous problem.

EXAMPLE 11 Assume that the ball in the previous example was being swung horizontally at a height of 2 meters above the ground so the center of the circle would be at $(0, 0, 2)$. Where would the ball be when it hit the ground?

Let $H(t) = (x(t), y(t), z(t))$ denote the position of the ball. When $t < 2.5$, we have $H(t) = (\cos(\pi t), \sin(\pi t), 2)$. After the point of release, the x- and y-components will be identical to what we had in the previous question.

The x and y components are given by $(0, 1) + (t - 2.5)(-\pi, 0) = (-\pi(t - 2.5), 1)$. Hence
$$x(t) = -\pi(t - 2.5)$$
and
$$y(t) = 1.$$

Gravity acts on the ball causing it to accelerate downward. We know that in Δt seconds the ball will fall $\frac{1}{2}g(\Delta t)^2$ meters, where g is the value of the acceleration due to gravity ($9.81\ m/s^2$).



Therefore, with $\Delta t = t - 2.5$, after the release of the ball the height will be $2 - \frac{1}{2}g(t - 2.5)^2$ meters. This tells us that the position of the ball between the time of release and when it hits the ground will be

$$H(t) = (-\pi(t - 2.5), 1, 2 - \frac{1}{2}g(t - 2.5)^2)$$

The ball will hit the ground when $2 - \frac{1}{2}g(t - 2.5)^2 = 0$. Solving for t we get

$$\frac{4}{g} = (t - 2.5)^2$$

or with $g = 9.81$

$$t = \sqrt{\frac{4}{9.81}} + 2.5 \cong 3.14$$

seconds.

To find the final position, we let $t = 3.14$ to get

$$H(3.14) = (-\pi(3.14 - 2.5), 1, 0) = (-2.01, 1, 0).$$

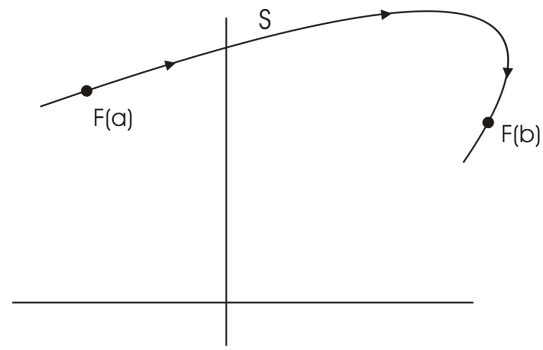
1.6 Arc Length of a Curve

We have seen how to use integration to find the length of a portion of the graph of a function. In this section, we will see how an almost identical method yields a formula for the length of a segment of a curve.

Problem:

Given a continuously differentiable vector-valued function $F(t) = (x(t), y(t))$, what is the length of the curve determined by $F(t)$ over the interval $[a, b]$?

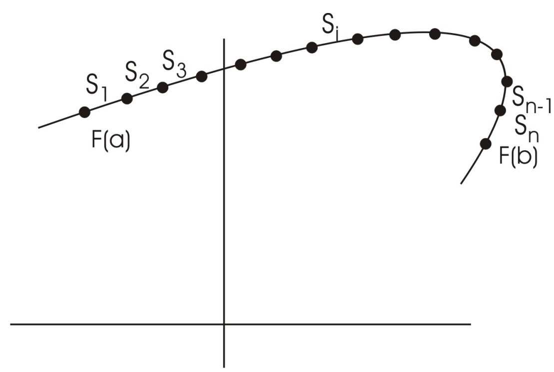
Let S denote the length of this segment of the curve.



We first partition $[a, b]$ with

$$P = \{a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b\}.$$

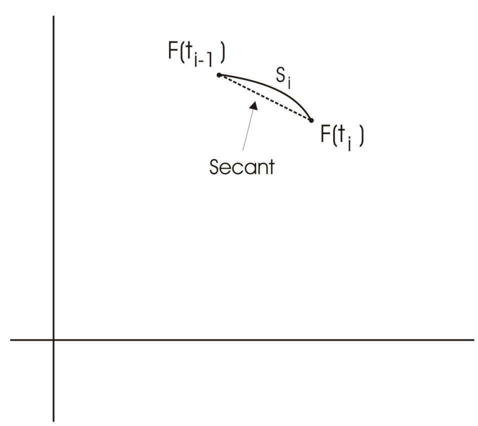
Let S_i denote the length of the arc joining $F(x_{i-1})$ and $F(x_i)$.



Then

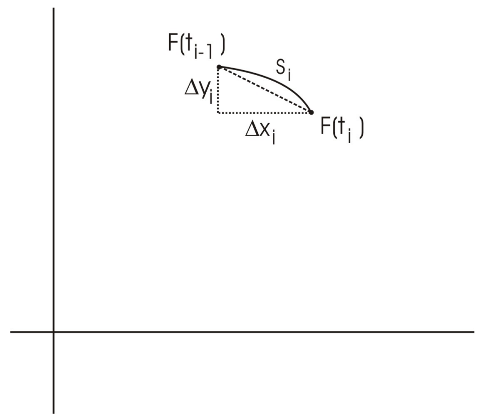
$$S = \sum_{i=1}^{n} S_i.$$

As was the case for single variable functions, if Δt_i is small, then S_i is approximately equal to the length of the secant line joining $F(x_{i-1})$ and $F(x_i)$.



It follows that

$$S_i \cong \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$



Using the incremental form for the linear approximation, we get

$$\Delta x_i \cong x'(t_i)\Delta t_i$$

and

$$\Delta y_i \cong y'(t_i)\Delta t_i.$$

Therefore,

$$\begin{aligned} S_i &\cong \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &\cong \sqrt{(x'(t_i)\Delta t_i)^2 + (y'(t_i)\Delta t_i)^2} \\ &= \sqrt{(x'(t_i))^2 + (y'(t_i))^2}\,\Delta t_i \end{aligned}$$

From this we get that

$$\begin{aligned} S &= \sum_{i=1}^{n} S_i \\ &\cong \sum_{i=1}^{n} \sqrt{(x'(t_i))^2 + (y'(t_i))^2}\,\Delta t_i \end{aligned}$$

The last sum is a Riemann sum for the function $\sqrt{(x'(t))^2 + (y'(t))^2}$. Letting $n \to \infty$, we get

$$S = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

We will make two useful observations.

1. Note that
$$\sqrt{(x'(t))^2 + (y'(t))^2} = \| F'(t) \|$$
If we think of $F(t)$ as representing the position of a particle, then arc length is just the distance traveled. Moreover, in this case, $\| F'(t) \|$ is the magnitude of the velocity of the particle. That is, $\| F'(t) \|$ is the speed of the particle. Therefore, the arc length formula tells us that
$$S = \int_a^b \| F'(t) \| \, dt$$
or
$$distance = \int_a^b (speed)\, dt$$
which is exactly what we would expect.

2. Suppose that $F(t) = (t, f(t))$. Then the arc length of the curve is just that length of the graph of $f(x)$ over the interval $[a, b]$. However, in this case, we have
$$x'(t) = 1$$
and
$$y'(t) = f'(t).$$
Therefore, the arc length formula for curves that we have just derived becomes
$$\begin{aligned} S &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \\ &= \int_a^b \sqrt{1^2 + (f'(t))^2}\, dt \\ &= \int_a^b \sqrt{1 + (f'(t))^2}\, dt. \end{aligned}$$
This agrees with the formula that we derived earlier for the arc length of the graph of a function.

EXAMPLE 12 A particle moves along a path so that its position is given by $F(t) = (t^2, t^3)$. Find the distance S the particle travels from $t = 0$ to $t = 1$.

In this case, $F'(t) = (2t, 3t^2)$ so that
$$\begin{aligned} \| F'(t) \| &= \sqrt{(2t)^2 + (3t^2)^2} \\ &= \sqrt{4t^2 + 9t^4} \\ &= |\, t \,| \sqrt{4 + 9t^2} \\ &= t\sqrt{4 + 9t^2} \end{aligned}$$

since $t \geq 0$ on $[0, 1]$.

It follows that

$$S = \int_0^1 t\sqrt{4+9t^2}\,dt.$$

To evaluate this integral we use the substitution $u = 4 + 9t^2$ so that $du = 18t\,dt$ and hence $t\,dt = \frac{du}{18}$. With this substitution, we have

$$\begin{aligned} S &= \int_0^1 t\sqrt{4+9t^2}\,dt \\ &= \int_4^{13} \frac{\sqrt{u}}{18}\,du \\ &= \frac{1}{18}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_4^{13} \\ &= \frac{1}{27}[(13)^{\frac{3}{2}} - (4)^{\frac{3}{2}}] \end{aligned}$$

◀

EXAMPLE 13 The curve $F(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$ is a spiral that turns in a counterclockwise direction around the origin. Note that $(0,0) = \lim\limits_{t\to\infty} F(t)$.

Find the total length of this spiral if it begins at $t = 0$.

In this case, we are viewing the curve over the interval $[0,\infty)$. Therefore, the arc length is the *improper integral*

$$S = \int_0^\infty \| F'(t) \| \; dt$$

To find $\| F'(t) \|$ note that

$$x'(t) = -e^{-t}\cos(t) - e^{-t}\sin(t)$$

so that

$$\begin{aligned} (x'(t))^2 &= (-e^{-t}\cos(t) - e^{-t}\sin(t))^2 \\ &= e^{-2t}\cos^2(t) + 2e^{-2t}\cos(t)\sin(t) + e^{-2t}\sin^2(t) \\ &= e^{-2t}(\cos^2(t) + \sin^2(t)) + 2e^{-2t}\cos(t)\sin(t) \\ &= e^{-2t} + 2e^{-2t}\cos(t)\sin(t). \end{aligned}$$

Similarly,

$$y'(t) = -e^{-t}\sin(t) + e^{-t}\cos(t)$$

so that

$$\begin{aligned}(y'(t))^2 &= (-e^{-t}\sin(t) + e^{-t}\cos(t))^2 \\ &= e^{-2t}\sin^2(t) - 2e^{-2t}\cos(t)\sin(t) + e^{-2t}\cos^2(t) \\ &= e^{-2t}(\sin^2(t) + \cos^2(t)) - 2e^{-2t}\cos(t)\sin(t) \\ &= e^{-2t} - 2e^{-2t}\cos(t)\sin(t).\end{aligned}$$

Therefore

$$(x'(t))^2 + (y'(t))^2 = 2e^{-2t}$$

and hence

$$\begin{aligned}\| F'(t) \| &= \sqrt{(x'(t))^2 + (y'(t))^2} \\ &= \sqrt{2e^{-2t}} \\ &= \sqrt{2}e^{-t}\end{aligned}$$

From here we get that

$$\begin{aligned}S &= \int_0^\infty \| F'(t) \| \, dt \\ &= \int_0^\infty \sqrt{2}e^{-t}\, dt \\ &= \lim_{b\to\infty} \int_0^b \sqrt{2}e^{-t}\, dt \\ &= \lim_{b\to\infty} -\sqrt{2}e^{-t} \Big|_0^b \\ &= \lim_{b\to\infty} -\sqrt{2}[e^{-b} - 1] \\ &= \sqrt{2}.\end{aligned}$$

We have shown that the total length of the spiral beginning at $t = 0$ is $\sqrt{2}$. ◀