Taylor’s Approximation Theorem

Created by

Barbara Forrest and Brian Forrest

Taylor's Remainder

Definition: [Taylor Remainder]

Assume that $f(x)$ is n times differentiable at $x = a$. Let

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

$R_{n,a}(x)$ is called the *n-th degree Taylor remainder function centered at* $x = a$.

Note: The error in using the Taylor polynomial to approximate $f(x)$ is given by

$$\textbf{Error} = | R_{n,a}(x) | .$$

Taylor's Remainder

Central Problem: Given a function $f(x)$ and a point $x = a$, how do we estimate the size of $R_{n,a}(x)$?

Theorem: [Taylor's Theorem]

Assume that $f(x)$ is $n+1$-times differentiable on an interval I containing $x = a$. Let $x \in I$. Then there exists a point c between x and a such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Examples

Example 1: Find $\lim_{x \to 0} \dfrac{\sin(x) - x}{x^2}$.

Solution: This is an indeterminate limit of the type $\frac{0}{0}$.

We know that if $f(x) = \sin(x)$, then

$$T_{1,0}(x) = T_{2,0}(x) = x.$$

Taylor's Theorem shows that for any $x \in [-1, 1]$, there exists a c between 0 and x such that

$$\mid \sin(x) - x \mid = \left| \frac{-\cos(c)}{3!} x^3 \right| \leq \frac{1}{6} \mid x \mid^3 \qquad (*)$$

since $\mid -\cos(c) \mid \leq 1$ regardless where c is located. So

$$-\frac{1}{6} \mid x \mid^3 \leq \sin(x) - x \leq \frac{1}{6} \mid x \mid^3 \qquad (**)$$

and then if $x \in [-1, 1]$ with $x \neq 0$,

$$-\frac{\mid x \mid}{6} \leq \frac{\sin(x) - x}{x^2} \leq \frac{\mid x \mid}{6}.$$

Hence, by the Squeeze Theorem,

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^2} = 0.$$

Taylor's Approximation Theorem

Important Remark: Suppose that $f^{(k+1)}(x)$ is a continuous function on $[-1, 1]$. Then so is the function

$$g(x) = \left| \frac{f^{(k+1)}(x)}{(k+1)!} \right|.$$

By Extreme Value Theorem there is an $M \geq 0$ such that

$$\left| \frac{f^{(k+1)}(x)}{(k+1)!} \right| \leq M \qquad (*)$$

for all $x \in [-1, 1]$. By Taylor's Theorem there is a c between x and 0 such that

$$| R_{k,0}(x) | = \left| \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1} \right|. \qquad (**)$$

Therefore,

$$\begin{aligned} | f(x) - T_{k,0}(x) | &= | R_{k,0}(x) | \\ &= \left| \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1} \right| \\ &\leq M \, | x |^{k+1} \end{aligned}$$

for all $x \in [-1, 1]$ since c is also in $[-1, 1]$.
It follows that

$$-M \, | x |^{k+1} \leq f(x) - T_{k,0}(x) \leq M \, | x |^{k+1} .$$

Taylor’s Approximation Theorem

Theorem: [Taylor’s Approximation Theorem]

Assume that $f^{(k+1)}(x)$ is continuous on $[-1, 1]$. Then there exists a constant $M \geq 0$ such that

$$| f(x) - T_{k,0}(x) | \leq M | x |^{k+1}$$

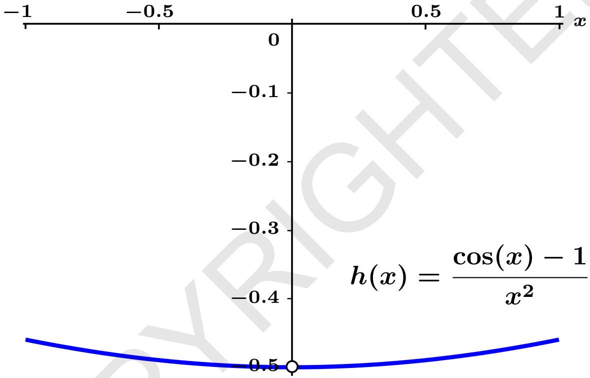
or equivalently that

$$-M | x |^{k+1} \leq f(x) - T_{k,0}(x) \leq M | x |^{k+1}$$

for each $x \in [-1, 1]$.

Remark: This theorem tells us that if $f^{(k+1)}(x)$ is continuous on $[-1, 1]$, then the error in using $T_{k,0}(x)$ to approximate $f(x)$ is of the same *order of magnitude* as $|x|^{k+1}$.

Examples



Example: Calculate

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2}.$$

Examples

Example (continued): Calculate

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2}.$$

We know that for $f(x) = \cos(x)$ we have

$$T_{2,0}(x) = 1 - \frac{x^2}{2}.$$

Moreover, all of the derivatives of $\cos(x)$ are continuous so the Taylor Approximation Theorem tells us that there is a constant M such that

$$-M \mid x \mid^3 \leq \cos(x) - (1 - \frac{x^2}{2}) \leq M \mid x \mid^3$$

for all $x \in [-1, 1]$.

Dividing by x^2 with $x \neq 0$ we have that

$$-M \mid x \mid \leq \frac{\cos(x) - (1 - \frac{x^2}{2})}{x^2} \leq M \mid x \mid$$

for all $x \in [-1, 1]$ with $x \neq 0$.

Examples

Example (continued):

Simplifying produces

$$-M \mid x \mid \leq \frac{\cos(x) - 1}{x^2} + \frac{1}{2} \leq M \mid x \mid$$

for all $x \in [-1, 1]$, $x \neq 0$.

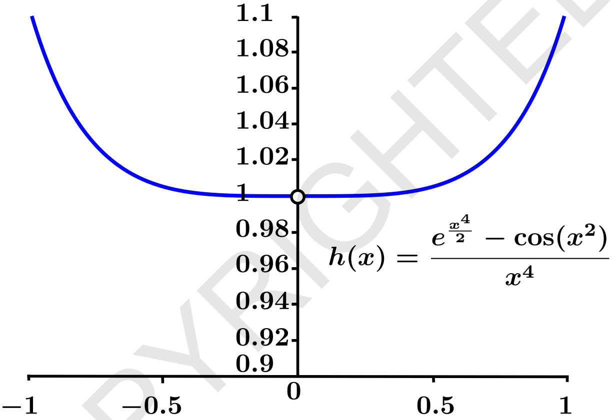
The Squeeze Theorem gives us

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} + \frac{1}{2} = 0$$

which is equivalent to

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} = -\frac{1}{2} .$$

Examples



Example: Find

$$\lim_{x \to 0} \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4}.$$

Examples

Example (continued): Find

$$\lim_{x\to 0}\frac{e^{\frac{x^4}{2}}-\cos(x^2)}{x^4}.$$

Solution: If $g(u)=e^u$, then $T_{1,0}(u)=L_0(u)=1+u$ so Taylor's Approximation Theorem says that there exits $M_1>0$ with

$$-M_1u^2\le e^u-(1+u)\le M_1u^2 \quad (*)$$

for all $u\in[-1,1]$.

If $x\in[-1,1]$, so is $u=\frac{x^4}{2}$. Then let $u=\frac{x^4}{2}$ to get

$$-M_1(\frac{x^4}{2})^2\le e^{\frac{x^4}{2}}-(1+\frac{x^4}{2})\le M_1(\frac{x^4}{2})^2$$

or

$$-\frac{M_1}{4}x^8\le e^{\frac{x^4}{2}}-(1+\frac{x^4}{2})\le\frac{M_1}{4}x^8 \quad (**)$$

for every $x\in[-1,1]$.

Examples

Example (continued): Find

$$\lim_{x\to 0} \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4}.$$

Solution (continued): We also can show that there exists a constant M_2 such that for any $v \in [-1, 1]$

$$-M_2 v^4 \le \cos(v) - (1 - \frac{v^2}{2}) \le M_2 v^4$$

since $1 - \frac{v^2}{2}$ is the third degree Taylor Polynomial for $\cos(v)$.

If $x \in [-1, 1]$, then so is x^2. If we let $v = x^2$, then we have

$$-M_2 x^8 \le \cos(x^2) - (1 - \frac{x^4}{2}) \le M_2 x^8 .$$

Multiplying by -1 gives

$$-M_2 x^8 \le (1 - \frac{x^4}{2}) - \cos(x^2) \le M_2 x^8. \quad (***)$$

Examples

Example (continued): Find

$$\lim_{x\to 0} \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4}.$$

Solution (continued): We have

$$-\frac{M_1}{4}x^8 \leq e^{\frac{x^4}{2}} - (1 + \frac{x^4}{2}) \leq \frac{M_1}{4}x^8 \quad (**)$$

and

$$-M_2x^8 \leq (1 - \frac{x^4}{2}) - \cos(x^2) \leq M_2x^8. \quad (***)$$

Adding the two inequalities together gives

$$-(\frac{M_1}{4} + M_2)x^8 \leq e^{\frac{x^4}{2}} - (\cancel{1} + \frac{x^4}{2}) + (\cancel{1} - \frac{x^4}{2}) - \cos(x^2) \leq (\frac{M_1}{4} + M_2)x^8$$

or

$$-(\frac{M_1}{4} + M_2)x^8 \leq [e^{\frac{x^4}{2}} - \cos(x^2)] - x^4 \leq (\frac{M_1}{4} + M_2)x^8$$

Examples

Example (continued): Find

$$\lim_{x\to 0} \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4}.$$

Solution (continued): Let $M = \frac{M_1}{4} + M_2$ and divide by x^4 to get

$$-Mx^4 \leq \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4} - 1 \leq Mx^4$$

for all $x \in [-1, 1]$ with $x \neq 0$.

Since

$$-Mx^4 \leq \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4} - 1 \leq Mx^4$$

the Squeeze Theorem shows that

$$\lim_{x\to 0} \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4} - 1 = 0$$

and hence that

$$\lim_{x\to 0} \frac{e^{\frac{x^4}{2}} - \cos(x^2)}{x^4} = 1.$$

Examples

Question: Is there an easier way to get this limit?