Big-O Notation:
Calculating Taylor Polynomials

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Taylor's Approximation Theorem II

Theorem: [Taylor's Approximation Theorem II]

Let $r > 0$. If $f(x)$ is $(n+1)$-times differentiable on $[-r, r]$ and $f^{(n+1)}(x)$ is continuous on $[-r, r]$, then

$$f(x) = T_{n,0}(x) + O(x^{n+1})$$

as $x \to 0$.

Question: We know that

$$\cos(x^2) - 1 = T_{7,0}(x) + O(x^8).$$

We can show that

$$\cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8).$$

Does this mean that

$$T_{7,0}(x) = -\frac{x^4}{2}?$$

Taylor's Approximation Theorem II

Theorem: [Characterization of Taylor Polynomials]

Assume that $r > 0$. Assume also that $f(x)$ is $(n+1)$-times differentiable on $[-r, r]$ and $f^{(n+1)}(x)$ is continuous on $[-r, r]$. If $p(x)$ is a polynomial of degree n or less with

$$f(x) = p(x) + O(x^{n+1}),$$

then $p(x) = T_{n,0}(x)$.

Example: We have that

$$\cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8)$$

so

$$T_{7,0}(x) = -\frac{x^4}{2}.$$

We also have

$$T_{3,0}(x) = 0 \quad \text{and} \quad T_{4,0}(x) = -\frac{x^4}{2}.$$

Taylor's Approximation Theorem II

Example: Let $f(x) = x^2(e^x - 1)\sin(x^2)$. Find $T_{5,0}(x)$ and in particular, find $f^{(4)}(0)$ and $f^{(5)}(0)$.

Solution: We know that

$$\sin(u) = u + O(u^3) \Rightarrow \sin(x^2) = x^2 + O(x^6).$$

Observe also that

$$e^x = 1 + x + O(x^2) \Rightarrow e^x - 1 = x + O(x^2).$$

Then

$$\begin{aligned} f(x) &= x^2(e^x - 1)\sin(x^2) = x^2(x + O(x^2))(x^2 + O(x^6)) \\ &= (x^3 + O(x^4))(x^2 + O(x^6)) \\ &= x^5 + O(x^9) + O(x^6) + O(x^{10}) \\ &= x^5 + O(x^6). \end{aligned}$$

The Characterization of Taylor Polynomials Theorem tells us that $x^5 = T_{5,0}(x)$. Since

$$T_{5,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5,$$

by matching coefficients, we get that $0 = \frac{f^{(4)}(0)}{4!}$ and $1 = \frac{f^{(5)}(0)}{5!}$. It follows that $f^{(4)}(0) = 0$ and $f^{(5)}(0) = 5! = 120$.

Big-O Arithmetic

Example: Find the 8th degree Taylor Polynomial for $f(x) = x^3 \sin(x)(e^{-x^2} - 1)$ centered at $x = 0$.

Question: How accurate do our estimates of $\sin(x)$ and $e^{-x^2} - 1$ have to be?

Solution: Because of the x^3 term we want the error in the product $\sin(x)(e^{-x^2} - 1)$ to be $O(x^6)$ or better since $x^3 O(x^6) = O(x^9)$ or better. Now

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots + O(x^k)$$

and

$$e^{-x^2} - 1 = -x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + O(x^j)$$

so the product $\sin(x)(e^{-x^2} - 1)$ will contain terms

$$xO(x^j) = O(x^{j+1})$$

and

$$-x^2 O(x^k) = O(x^{k+2}).$$

Therefore we must have

$$j + 1 \geq 6 \Rightarrow j \geq 5 \quad \text{and} \quad k + 2 \geq 6 \Rightarrow k \geq 4.$$

We could try

$$\sin(x) = x - \frac{x^3}{6} + O(x^4) \quad \text{and} \quad e^{-x^2} - 1 = -x^2 + \frac{x^4}{2} + O(x^5).$$

Big-O Arithmetic

Example: Find the 8th degree Taylor Polynomial for $f(x) = x^3 \sin(x)(e^{-x^2} - 1)$ centered at $x = 0$.

Solution (continued): We have

$$\begin{aligned} \sin(x)(e^{-x^2} - 1) &= (x - \frac{x^3}{6} + O(x^4))(-x^2 + \frac{x^4}{2} + O(x^5)) \\ &= -x^3 + \frac{x^5}{2} + \frac{x^5}{6} + O(x^6) \\ &= -x^3 + \frac{2}{3}x^5 + O(x^6) \end{aligned}$$

and hence

$$x^3 \sin(x)(e^{-x^2} - 1) = -x^6 + \frac{2}{3}x^8 + O(x^9).$$

This shows

$$T_{8,0}(x) = -x^6 + \frac{2}{3}x^8.$$

Big-O Arithmetic

Example: If $f(x) = x^3 \sin(x)(e^{-x^2} - 1)$, find $f^{(5)}(0)$ and $f^{(8)}(0)$.

Solution: We know that the coefficient a_k of x^k in the Taylor Polynomial is $\dfrac{f^{(k)}(0)}{k!}$ so

$$f^{(k)}(0) = a_k \cdot k!$$

Since

$$T_{8,0}(x) = -x^6 + \frac{2}{3}x^8$$

we have

$$f^{(5)}(0) = a_5 \cdot 5! = 0 \cdot 5! = 0$$

and

$$f^{(8)}(0) = a_8 \cdot 8! = \frac{2}{3} \cdot 8!$$