Big-O Notation

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Motivation

Suppose we know that

$$\lim_{x \to 0} f(x) = 0.$$

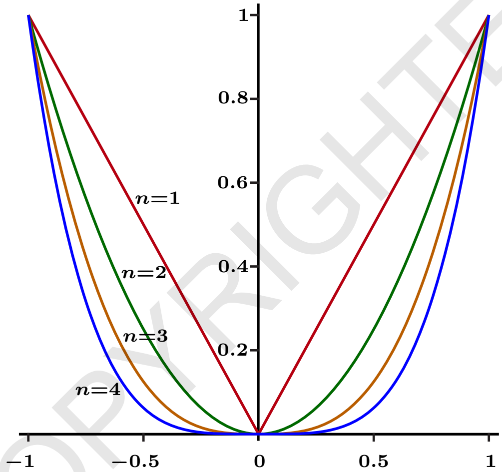
Question: How quickly does $f(x)$ approach 0?

For example, consider

$$\lim_{x \to 0} x^2 = 0 = \lim_{x \to 0} |x^{17}|.$$

Which function approaches the limit $L = 0$ more quickly?

The Behavior of $|x|^n$



Example: Consider the functions $| \, x^n \, |$. The larger the value of n, the faster x^n approaches 0.

Big-O Notation

Definition: [Big-O Notation]

We say that $f(x)$ *is Big-O of* $g(x)$ *as* $x \to a$ if there exists an $\epsilon > 0$ and an $M > 0$ such that

$$|f(x)| \leq M|g(x)|$$

for all $x \in (a - \epsilon, a + \epsilon)$ except possibly at $x = a$.

In this case, we write

$$f(x) = O(g(x)) \quad \text{as } x \to a$$

or simply $f(x) = O(g(x))$ if a is understood.

If $f(x)$ is Big-O of $g(x)$ as $x \to a$, then we say that $f(x)$ has *order of magnitude that is less than or equal to that of* $g(x)$ *near* $x = a$.

$O(x^n)$

Question: What do we know if

$$f(x) = O(x^n)?$$

This means that there exists an $M > 0$ so that

$$-M|x^n| \leq f(x) \leq M|x^n|$$

for all $x \in (-\epsilon, \epsilon)$ except possibly at $x = 0$. Since

$$\lim_{x \to 0} -M|x^n| = 0 = \lim_{x \to 0} M|x^n|$$

the Squeeze Theorem shows that

$$\lim_{x \to 0} f(x) = 0.$$

Big-O Notation

Definition: [Extended Big-O Notation]

Suppose that $f(x)$, $g(x)$ and $h(x)$ are defined on an open interval containing $x = a$, except possibly at $x = a$. We write

$$f(x) = g(x) + O(h(x)) \text{ as } x \to a$$

if

$$f(x) - g(x) = O(h(x)) \text{ as } x \to a.$$

We may omit the $x \to a$ condition if a is understood.

Remark: The notation $f(x) = g(x) + O(h(x))$ tells us that $f(x) \approx g(x)$ near $x = a$ with an error that is an order of magnitude at most that of $h(x)$.

The Behavior of $\sin(x)$

Example: Consider $f(x) = \sin(x)$. By Taylor's Theorem we get that if $x \in [-1, 1]$, there exists some c between x and 0 such that

$$|\sin(x) - T_{1,0}(x)| = |\sin(x) - x| = \left|\frac{f''(c)}{2!}x^2\right| = \left|\frac{-\sin(c)}{2}x^2\right| \leq \frac{1}{2}|x^2|.$$

Hence,

$$\sin(x) - x = O(x^2),$$

so that

$$\sin(x) = x + O(x^2).$$

That is, $T_{1,0}(x) = L_0(x) = x$ approximates the function $f(x) = \sin(x)$ near $x = 0$ with an error that is an order of magnitude of at most x^2.

Since $T_{1,0}(x) = T_{2,0}(x)$ for $\sin(x)$, we can interpret x as $T_{2,0}(x)$ instead of $T_{1,0}(x)$. If we apply Taylor's Theorem again using $T_{2,0}(x)$, we get that

$$\sin(x) = x + O(x^3).$$

This is a stronger statement because x^3 is an order of magnitude smaller than x^2 near $x = 0$ and as such the approximation $\sin(x) \cong x$ is even better than was suggested above.

Taylor's Approximation Theorem II

Theorem: [Taylor's Approximation Theorem II]
Let $r > 0$. If $f(x)$ is $(n+1)$-times differentiable on $[-r, r]$ and $f^{(n+1)}(x)$ is continuous on $[-r, r]$, then

$$f(x) = T_{n,0}(x) + O(x^{n+1})$$

as $x \to 0$.

Proof: By the Extreme Value Theorem, M can be chosen so that

$$|f^{(n+1)}(x)| \leq M$$

for all $x \in [-r, r]$.

Taylor's Theorem implies that for any $x \in [-r, r]$, there exists a c between x and 0 so that

$$|f(x) - T_{n,0}(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \leq \left| \frac{M}{(n+1)!} x^{n+1} \right| = \frac{M}{(n+1)!} |x^{n+1}|.$$

Therefore,

$$f(x) - T_{n,0}(x) = O(x^{n+1}) \Rightarrow f(x) = T_{n,0}(x) + O(x^{n+1})$$

as $x \to 0$.