Big-O Notation: Examples

Created by

Barbara Forrest and Brian Forrest

Big-O Arithmetic

Theorem: [Arithmetic of Big-O]

Assume that $f(x) = O(x^n)$ and $g(x) = O(x^m)$ as $x \to 0$, for some $m, n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then we have the following.

1) $c(O(x^n)) = O(x^n)$. That is, $(cf)(x) = c \cdot f(x) = O(x^n)$.
2) $O(x^n) + O(x^m) = O(x^k)$, where $k = \min\{n, m\}$. That is, $f(x) \pm g(x) = O(x^k)$.
3) $O(x^n)O(x^m) = O(x^{n+m})$. That is, $f(x)g(x) = O(x^{n+m})$.
4) If $k \leq n$, then $f(x) = O(x^k)$.
5) If $k \leq n$, then $\frac{1}{x^k}O(x^n) = O(x^{n-k})$. That is, $\frac{f(x)}{x^k} = O(x^{n-k})$.
6) $f(u^k) = O(u^{kn})$. That is, we can simply substitute $x = u^k$.

The Fundamental Trig Limit

Example: The Fundamental Trig Limit states that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

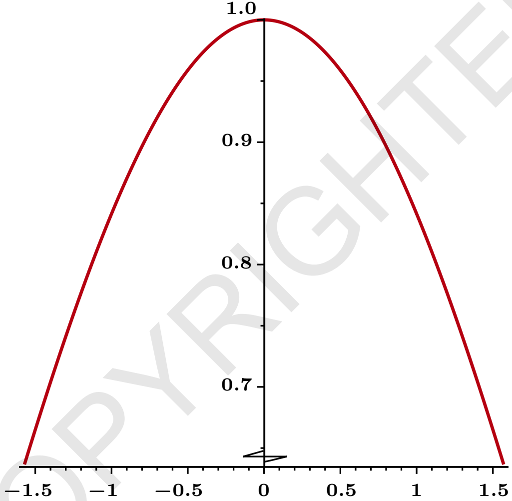
This can be explained using the Big-O notation as follows:

$$\sin(x) = x + O(x^3)$$

so

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(x)}{x} &= \lim_{x \to 0} \frac{x + O(x^3)}{x} \\ &= \lim_{x \to 0} 1 + O(x^2) \\ &= 1. \end{aligned}$$

The Fundamental Trig Limit



Example: Let

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

The Fundamental Trig Limit

Claim:

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

is differentiable at $x = 0$ with $f'(0) = 0$.

We want

$$f'(0) = \lim_{h \to 0} \frac{\frac{\sin(h)}{h} - 1}{h}.$$

Now $\sin(h) = h + O(h^3) \Rightarrow \frac{\sin(h)}{h} = 1 + O(h^2)$.

It follows that

$$\begin{aligned} \lim_{h \to 0} \frac{\frac{\sin(h)}{h} - 1}{h} &= \lim_{h \to 0} \frac{O(h^2)}{h} \\ &= \lim_{h \to 0} O(h) \\ &= 0. \end{aligned}$$

Examples

Example: Show that $f(x) = \cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8)$.

Solution: We begin by observing that if $g(u) = \cos(u)$, then since the third degree Taylor polynomial for $g(u)$ centered at $u = 0$ is

$$T_{3,0}(u) = 1 - \frac{u^2}{2}$$

the Taylor Approximation Theorem II gives us that

$$g(u) = 1 - \frac{u^2}{2} + O(u^4).$$

Arithmetic Rule (6) allows us to substitute x^2 for u to get

$$g(x^2) = \cos(x^2) = 1 - \frac{(x^2)^2}{2} + O((x^2)^4) = 1 - \frac{x^4}{2} + O(x^8).$$

Then

$$f(x) = \cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8).$$

Examples

Example: Evaluate

$$\lim_{x\to 0} \frac{\cos(x^2) - 1}{x^4}.$$

Solution: We saw that

$$\cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8).$$

Hence

$$\begin{aligned} \lim_{x\to 0} \frac{\cos(x^2) - 1}{x^4} &= \lim_{x\to 0} \frac{-\frac{x^4}{2} + O(x^8)}{x^4} \\ &= \lim_{x\to 0} -\frac{1}{2} + O(x^4) \\ &= -\frac{1}{2}. \end{aligned}$$

Examples

Example: Evaluate

$$\lim_{x\to 0} \frac{x^2 \sin(x^2)(e^x - 1)}{(\cos(x) - 1)(\sin^2(x))(\sin(2x))}.$$

Solution: Using Big-O arithmetic observe that

$$\sin(u) = u + O(u^3),$$

so

$$\sin(x^2) = x^2 + O(x^6).$$

Next, observe that

$$e^x = 1 + x + O(x^2)$$

so

$$e^x - 1 = x + O(x^2).$$

Then

$$\begin{aligned} x^2(e^x - 1)\sin(x^2) &= x^2(x + O(x^2))(x^2 + O(x^6)) \\ &= (x^3 + O(x^4))(x^2 + O(x^6)) \\ &= x^5 + O(x^9) + O(x^6) + O(x^{10}) \\ &= x^5 + O(x^6). \end{aligned}$$

Examples

Example: Evaluate

$$\lim_{x\to 0} \frac{x^2 \sin(x^2)(e^x - 1)}{(\cos(x) - 1)(\sin^2(x))(\sin(2x))}.$$

Solution (continued): Next $\cos(x) = 1 - \frac{x^2}{2} + O(x^4)$ and so

$$\cos(x) - 1 = -\frac{x^2}{2} + O(x^4).$$

We have

$$\sin(u) = u + O(u^3) \Rightarrow \sin(2x) = 2x + O(x^3)$$

and

$$\begin{aligned}\sin^2(x) &= (x + O(x^3))(x + O(x^3)) \\ &= x^2 + O(x^4) + O(x^4) + O(x^6) \\ &= x^2 + O(x^4).\end{aligned}$$

Therefore

$$\begin{aligned}(\cos(x) - 1)(\sin^2(x)) &= \left(\frac{-x^2}{2} + O(x^4)\right)(x^2 + O(x^4)) \\ &= -\frac{x^4}{2} + O(x^6) + O(x^6) + O(x^8) \\ &= -\frac{x^4}{2} + O(x^6).\end{aligned}$$

Examples

Example: Evaluate

$$\lim_{x\to 0} \frac{x^2 \sin(x^2)(e^x - 1)}{(\cos(x) - 1)(\sin^2(x))(\sin(2x))}.$$

Solution (continued): Next we get

$$\begin{aligned}(\cos(x) - 1)(\sin^2(x))(\sin(2x)) &= \left(-\frac{x^4}{2} + O(x^6)\right)(2x + O(x^3)) \\ &= -x^5 + O(x^7) + O(x^7) + O(x^9) \\ &= -x^5 + O(x^7).\end{aligned}$$

Finally

$$\begin{aligned}\lim_{x\to 0} \frac{x^2 \sin(x^2)(e^x - 1)}{(\cos(x) - 1)(\sin^2(x))(\sin(2x))} &= \lim_{x\to 0} \frac{x^5 + O(x^6)}{-x^5 + O(x^7)} \\ &= \lim_{x\to 0} \frac{1 + O(x)}{-1 + O(x^2)} \\ &= -1.\end{aligned}$$

Examples

Example: Evaluate

$$\lim_{x \to 0} \frac{x^2 \sin(x^2)(e^x - 1)}{(\cos(x) - 1)(\sin^2(x))(\sin(2x))}.$$

Heuristic Solution:

$$\begin{aligned} \frac{x^2 \sin(x^2)(e^x - 1)}{(\cos(x) - 1)(\sin^2(x))(\sin(2x))} &\cong \frac{x^2 \cdot x^2 \cdot x}{-\frac{x^2}{2} \cdot x^2 \cdot 2x} \\ &= \frac{x^5}{-x^5} \\ &= -1. \end{aligned}$$