# Arithmetic with Big-O Notation

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# Big-O Notation

**Definition: [Big-O Notation]**

We say that $f(x)$ *is Big-O of* $g(x)$ *as* $x \to a$ if there exists an $\epsilon > 0$ and an $M > 0$ such that

$$|f(x)| \leq M|g(x)|$$

for all $x \in (a - \epsilon, a + \epsilon)$ except possibly at $x = a$.

In this case, we write

$$f(x) = O(g(x)) \quad \text{as } x \to a.$$

We can write

$$f(x) = g(x) + O(h(x)) \text{ as } x \to a$$

if

$$f(x) - g(x) = O(h(x)) \text{ as } x \to a.$$

# Big-O Notation

**Theorem: [Taylor's Approximation Theorem II]**

Let $r > 0$. If $f(x)$ is $(n+1)$-times differentiable on $[-r, r]$ and $f^{(n+1)}(x)$ is continuous on $[-r, r]$, then

$$f(x) = T_{n,0}(x) + O(x^{n+1})$$

as $x \to 0$.

# Big-O Notation

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as $x \to 0$.

**Question:** Assume that

$$f(x) = O(x^2) \text{ as } x \to 0$$

and

$$g(x) = O(x^3) \text{ as } x \to 0.$$

What can we say about

$$h(x) = f(x) + g(x)?$$

# Big-O and Sums

**Observation:** Assume that

$$f(x) = O(x^2), \quad g(x) = O(x^3) \text{ as } x \to 0.$$

Then we can find an $0 < \epsilon \leq 1$ and two constants $M_1, M_2 > 0$ so that

$$|f(x)| \leq M_1 |x^2|$$

and

$$|g(x)| \leq M_2 |x^3|$$

for all $x \in [-\epsilon, \epsilon]$, except possibly at $x = 0$.

If $x \in [-\epsilon, \epsilon]$ with $x \neq 0$, then

$$\begin{aligned} |f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq M_1 |x^2| + M_2 |x^3| \\ &\leq M_1 |x^2| + M_2 |x^2| \\ &= (M_1 + M_2) |x^2| \end{aligned}$$

Hence

$$f(x) + g(x) = O(x^2) \text{ as } x \to 0.$$

# Big-O and Sums

**Observation:** In general, if

$$f(x) = O(x^n) \quad \text{and} \quad g(x) = O(x^m) \quad \text{as } x \to 0,$$

then

$$f(x) + g(x) = O(x^k) \text{ as } x \to 0$$

where

$$k = \min\{n, m\}.$$

That is, *the potential error in a sum is at least as large as the error in either part.*

We write

$$O(x^n) + O(x^m) = O(x^k)$$

where

$$k = \min\{n, m\}.$$

# Big-O Arithmetic

**Theorem: [Arithmetic of Big-O]**

Assume that $f(x) = O(x^n)$ and $g(x) = O(x^m)$ as $x \to 0$, for some $m, n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then we have the following:

1) $c(O(x^n)) = O(x^n)$. That is, $(cf)(x) = c \cdot f(x) = O(x^n)$.
2) $O(x^n) + O(x^m) = O(x^k)$, where $k = \min\{n, m\}$. That is, $f(x) \pm g(x) = O(x^k)$.
3) $O(x^n)O(x^m) = O(x^{n+m})$. That is, $f(x)g(x) = O(x^{n+m})$.
4) If $k \le n$, then $f(x) = O(x^k)$.
5) If $k \le n$, then $\frac{1}{x^k}O(x^n) = O(x^{n-k})$. That is, $\frac{f(x)}{x^k} = O(x^{n-k})$.
6) $f(u^k) = O(u^{kn})$. That is, we can simply substitute $x = u^k$.

**Note:** In fact (5) is true if we replace $x^k$ by $x^\alpha$ for any $\alpha \in \mathbb{R}$.

# Big-O Arithmetic

**Example:** Show that $f(x) = \cos(x^2) - 1 = -\frac{x_4}{2} + O(x^8)$. Use this result to evaluate

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}.$$

**Solution:** We begin by observing that if $g(u) = \cos(u)$, then since the third degree Taylor polynomial for $g(u)$ centered at $u = 0$ is

$$T_{3,0}(u) = 1 - \frac{u^2}{2}$$

Taylor's Approximation Theorem II gives us that

$$g(u) = 1 - \frac{u^2}{2} + O(u^4).$$

Arithmetic Rule (6) allows us to substitute $x^2$ for $u$ to get

$$\cos(x^2) = g(x^2) = 1 - \frac{(x^2)^2}{2} + O((x^2)^4) = 1 - \frac{x^4}{2} + O(x^8).$$

Then

$$f(x) = \cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8).$$

# Big-O Arithmetic

**Example (continued):** Show that $f(x) = \cos(x^2) - 1 = -\frac{x_4}{2} + O(x^8)$.
Use this result to evaluate

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}.$$

**Solution (continued):**
To evaluate

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}$$

we use the Arithmetic Rules to get

$$\begin{aligned} \lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4} &= \lim_{x \to 0} \frac{-\frac{x^4}{2} + O(x^8)}{x^4} \\ &= \lim_{x \to 0} -\frac{1}{2} + O(x^4) \\ &= -\frac{1}{2} \end{aligned}$$

since

$$\lim_{x \to 0} O(x^n) = 0$$

for every $n > 0$.

# Big-O Arithmetic

**Example:** Let $f(x) = \sin(x)(e^{-x^2} - 1)$. Show that

$$f(x) = -x^3 + O(x^5).$$

**Solution:** We know that

$$\sin(x) = x + O(x^3)$$

and that

$$e^u = 1 + u + O(u^2) \Rightarrow e^u - 1 = u + O(u^2) \Rightarrow e^{-x^2} - 1 = -x^2 + O(x^4)$$

since

$$O((-x^2)^2) = O(x^4).$$

Therefore using the Arithmetic Rules for Big-O:

$$\begin{aligned}
\sin(x)(e^{-x^2} - 1) &= (x + O(x^3))(-x^2 + O(x^4)) \\
&= -x^3 + xO(x^4) + (-x^2)O(x^3) + O(x^3)O(x^4) \\
&= -x^3 + O(x^5) + O(x^5) + O(x^7) \\
&= -x^3 + O(x^5).
\end{aligned}$$

# Big O Arithmetic

**Important Remark:** Suppose

$$f(x) = 1 - x^2 + O(x^4)$$

and

$$g(x) = x + O(x^2).$$

Then

$$f(x)g(x) = (1 - x^2 + O(x^4))(x + O(x^2))$$

$$= 1 \cdot x + 1 \cdot O(x^2) - x^2 \cdot x - x^2 \cdot O(x^2) + x \cdot O(x^4) + O(x^4) \cdot O(x^2)$$

$$= x + O(x^2) - x^3 + O(x^4) + O(x^5) + O(x^6)$$

$$= x + O(x^2) + O(x^3) + O(x^4) + O(x^5) + O(x^6)$$

$$= x + O(x^2)$$

since once we have the term $O(x^2)$, all higher degree terms (such as $x^3$) do not add any additional accuracy to the estimate.