Arithmetic with Big-O Notation
Part 2

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Big-O and Sums

Recall: If,

$$f(x) = O(x^n) \quad \text{and} \quad g(x) = O(x^m) \quad \text{as } x \to 0,$$

then

$$f(x) + g(x) = O(x^k) \text{ as } x \to 0$$

where

$$k = \min\{n, m\}.$$

We write

$$O(x^n) + O(x^m) = O(x^k)$$

where

$$k = \min\{n, m\}.$$

Big-O Arithmetic

Theorem: [Arithmetic of Big-O]

Assume that $f(x) = O(x^n)$ and $g(x) = O(x^m)$ as $x \to 0$, for some $m, n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then we have the following:

1) $c(O(x^n)) = O(x^n)$. That is, $(cf)(x) = c \cdot f(x) = O(x^n)$.
2) $O(x^n) + O(x^m) = O(x^k)$, where $k = \min\{n, m\}$. That is, $f(x) \pm g(x) = O(x^k)$.
3) $O(x^n)O(x^m) = O(x^{n+m})$. That is, $f(x)g(x) = O(x^{n+m})$.
4) If $k \leq n$, then $f(x) = O(x^k)$.
5) If $k \leq n$, then $\frac{1}{x^k}O(x^n) = O(x^{n-k})$. That is, $\frac{f(x)}{x^k} = O(x^{n-k})$.
6) $f(u^k) = O(u^{kn})$. That is, we can simply substitute $x = u^k$.

Note: In fact (5) is true if we replace x^k by x^α for any $\alpha \in \mathbb{R}$.

Big-O Arithmetic

Example: Show that $f(x) = \cos(x^2) - 1 = -\frac{x_4}{2} + O(x^8)$. Use this result to evaluate

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}.$$

Solution: We begin by observing that if $g(u) = \cos(u)$, then since the third degree Taylor polynomial for $g(u)$ centered at $u = 0$ is

$$T_{3,0}(u) = 1 - \frac{u^2}{2}$$

Taylor's Approximation Theorem II gives us that

$$g(u) = 1 - \frac{u^2}{2} + O(u^4).$$

Arithmetic Rule (6) allows us to substitute x^2 for u to get

$$\cos(x^2) = g(x^2) = 1 - \frac{(x^2)^2}{2} + O((x^2)^4) = 1 - \frac{x^4}{2} + O(x^8).$$

Then

$$f(x) = \cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8).$$

Big-O Arithmetic

Example (continued): Show that $f(x) = \cos(x^2) - 1 = -\frac{x_4}{2} + O(x^8)$.
Use this result to evaluate

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}.$$

Solution (continued):
To evaluate

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}$$

we use the Arithmetic Rules to get

$$\begin{aligned} \lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4} &= \lim_{x \to 0} \frac{-\frac{x^4}{2} + O(x^8)}{x^4} \\ &= \lim_{x \to 0} -\frac{1}{2} + O(x^4) \\ &= -\frac{1}{2} \end{aligned}$$

since

$$\lim_{x \to 0} O(x^n) = 0$$

for every $n > 0$.

Big-O Arithmetic

Example: Let $f(x) = \sin(x)(e^{-x^2} - 1)$. Show that

$$f(x) = -x^3 + O(x^5).$$

Solution: We know that

$$\sin(x) = x + O(x^3)$$

and that

$$e^u = 1+u+O(u^2) \Rightarrow e^u-1 = u+O(u^2) \Rightarrow e^{-x^2}-1 = -x^2+O(x^4)$$

since

$$O((-x^2)^2) = O(x^4).$$

Therefore using the Arithmetic Rules for Big-O:

$$\begin{aligned}
\sin(x)(e^{-x^2} - 1) &= (x + O(x^3))(-x^2 + O(x^4)) \\
&= -x^3 + xO(x^4) + (-x^2)O(x^3) + O(x^3)O(x^4) \\
&= -x^3 + O(x^5) + O(x^5) + O(x^7) \\
&= -x^3 + O(x^5).
\end{aligned}$$

Big O Arithmetic

Important Remark: Suppose

$$f(x) = 1 - x^2 + O(x^4)$$

and

$$g(x) = x + O(x^2).$$

Then

$$f(x)g(x) = (1 - x^2 + O(x^4))(x + O(x^2))$$

$$= 1 \cdot x + 1 \cdot O(x^2) - x^2 \cdot x - x^2 \cdot O(x^2) + x \cdot O(x^4) + O(x^4) \cdot O(x^2)$$

$$= x + O(x^2) - x^3 + O(x^4) + O(x^5) + O(x^6)$$

$$= x + O(x^2) + O(x^3) + O(x^4) + O(x^5) + O(x^6)$$

$$= x + O(x^2)$$

since once we have the term $O(x^2)$, all higher degree terms (such as x^3) do not add any additional accuracy to the estimate.