Applications of the MVT: The Second Derivative Test

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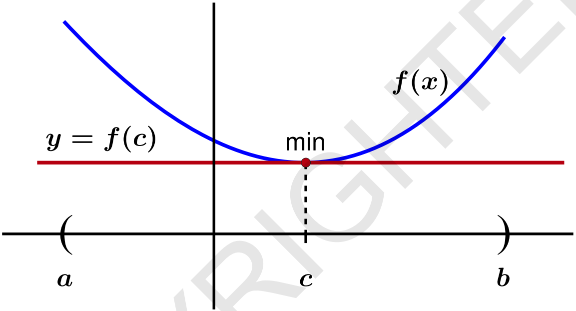
Critical Points and Concavity



Problem: Given a critical point c for $f(x)$ with $f'(c) = 0$, does concavity help us determine whether c is either a local maximum or a local minimum?

1) If $f(x)$ is concave downwards on an interval containing $x = c$, then $(c, f(c))$ should be a local maximum.
2) If $f(x)$ is concave upwards on an interval containing $x = c$, then $(c, f(c))$ should be a local minimum.

The Second Derivative Test:

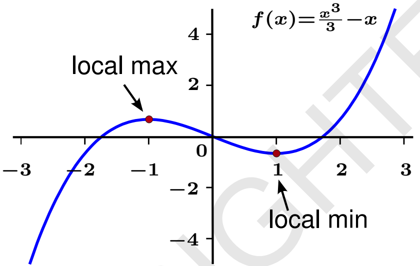
Theorem: [Second Derivative Test]

Assume that $f'(c) = 0$ and that $f''(x)$ is continuous at $x = c$.

i) If $f''(c) < 0$, then $f(x)$ has a local maximum at c.

ii) If $f''(c) > 0$, then $f(x)$ has a local minimum at c.

The Second Derivative Test



Example: Let

$$f(x) = \frac{x^3}{3} - x.$$

Then

$$f'(x) = x^2 - 1$$

so the critical points are $x = -1$ and $x = 1$.

We have $f''(x) = 2x$. Since

i) $f''(-1) < 0$, then $f(x)$ has a local maximum at $x = -1$.

ii) $f''(1) > 0$, then $f(x)$ has a local minimum at $x = 1$.

Extreme Value Theorem

Theorem: [The Extreme Value Theorem (EVT)]

Suppose that $f(x)$ is continuous on $[a, b]$. There exists c_1 and $c_2 \in [a, b]$ such that

$$f(c_1) \leq f(x) \leq f(c_2)$$

for all $x \in [a, b]$.

Extreme Value Theorem

Observation: The EVT ensures that a continuous function $f(x)$ defined on a closed interval $[a, b]$ achieves its global maximum and global minimum on $[a, b]$, but it does not tell us how to find these values.

Important Fact: Assume that $f(x)$ has either a global maximum or global minimum at $c \in [a, b]$. Then either

1) c is an endpoint of the interval $[a, b] \Rightarrow c = a$ or $c = b$

or

2) c is **not** an endpoint of the interval $[a, b] \Rightarrow c \in (a, b)$ is a local maximum or local minimum $\Rightarrow$ c is a critical point.

Global Maxima and Global Minima

Summary [Finding the Global Maximum and Global Minimum]

To find the maximum and minimum for a continuous function $f(x)$ on $[a, b]$:

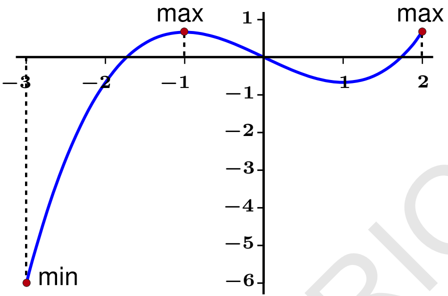
Step 1: Evaluate $f(a)$ and $f(b)$.

Step 2: Find all critical points c in (a, b) such that $f'(c) = 0$ and $f'(c)$ does not exist, where applicable.

Step 3: Evaluate the function at each of the critical points.

Step 4: The global maximum is at the point that produces the *largest* value from Steps 1 and 3. The global minimum is at the point that produces the *smallest* value from Steps 1 and 3.

Global Maxima and Global Minima



Example: Let

$$f(x) = \frac{x^3}{3} - x.$$

Find the maximum and minimum values of $f(x)$ on $[-3, 2]$.

Solution:

Step 1: Evaluate endpoints: $f(-3) = -6$ and $f(2) = \frac{2}{3}$.

Step 2: Since $f'(x) = x^2 - 1$ the critical points are $x = \pm 1$.

Step 3: $f(-1) = \frac{2}{3}$ and $f(1) = -\frac{2}{3}$.

Step 4: The global maximum is at both $x = -1$ and $x = 2$ where the maximum value is $\frac{2}{3}$. The global minimum is at $x = -3$ where the minimum value is -6.