Mean Value Theorem

Created by

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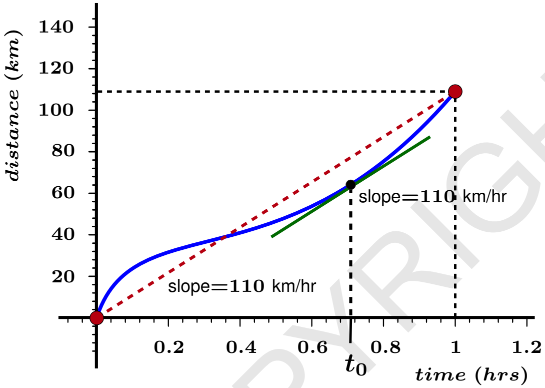
A Problem

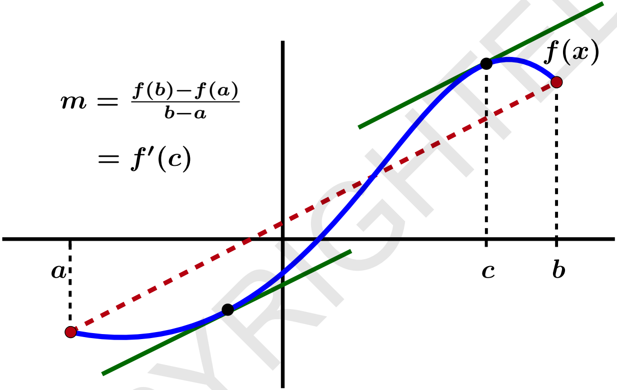


Problem: A car travels forward a distance of 110 km in one hour along a road with a posted speed limit of 100 km/hr. Prove that at some point in the journey the car was speeding.

Average velocity $= 110$ km/hr.

At t_0 the tangent line is parallel to the secant line $\Rightarrow v(t_o) = 110$ km/hr.

Generic Case

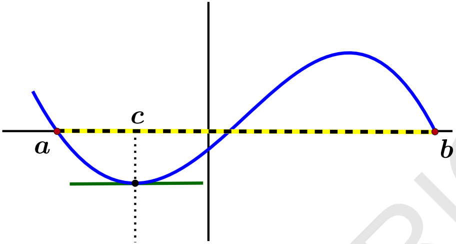


Question: Is there always a point at which the *instantaneous* rate of change equals the *average* rate of change?

Average rate of change $= \frac{f(b)-f(a)}{b-a}$

$$= f'(c)?$$

Rolle’s Theorem



Theorem: [Rolle’s Theorem]

Assume that $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and that $f(a) = 0 = f(b)$.

Then there exists a $c \in (a, b)$ with

$$f'(c) = 0.$$

Proof:

Case 1: $f(x) = 0$ for all $x \in [a, b] \Rightarrow f'(c) = 0$ for any $c \in (a, b)$.

Case 2: $f(x) \neq 0$ for some $x \in [a, b]$.

Then $f(x)$ either attains its maximum at some point $c \in (a, b)$ and $f'(c) = 0$, or $f(x)$ attains its minimum at some point $c \in (a, b)$ and $f'(c) = 0$.

Mean Value Theorem

Theorem: [Mean Value Theorem]

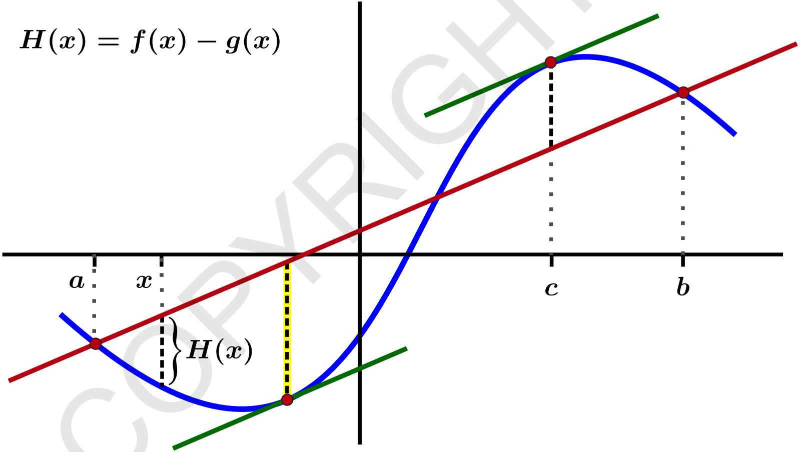
Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometric Proof

- $f(x)$
- $g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$$H(x) = f(x) - g(x)$$



Analytic Proof

Let

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \Rightarrow g'(x) = \frac{f(b) - f(a)}{b - a}$$

for all $x \in (a, b)$. Let $H(x) = f(x) - g(x)$. Then

$$H(a) = f(a) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(a - a)\right) = f(a) - f(a) = 0.$$

$$\begin{aligned} H(b) &= f(b) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(b - a)\right) \\ &= f(b) - (f(a) + f(b) - f(a)) \\ &= 0. \end{aligned}$$

Rolle's Theorem $\Rightarrow$ there exists a $c \in (a, b)$ such that

$$\begin{aligned} 0 = H'(c) = f'(c) - g'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ &\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

MVT Problem

Problem: Two runners compete in a 100 m race. If they finish the race in exactly the same amount of time t_f, show that there exists some time $c \in (0, t_f)$ such that at time c the two runners were traveling exactly the same speed.