# L'Hôpital's Rule

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# L'H$\hat{o}$pital's Rule

**Recall:** If $h(x) = \frac{f(x)}{g(x)}$ and if

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

then we do *not* know whether or not $\lim_{x \to a} h(x)$ exists.

For this reason, we call such a situation an *indeterminate form of type* $\frac{0}{0}$.

Similarly, if

$$\lim_{x \to a} f(x) = \pm\infty = \lim_{x \to a} g(x),$$

we would not be able to determine immediately if the limit of the quotient exists.

We call this situation an *indeterminate form of type* $\frac{\infty}{\infty}$.

**Note:** *L'Hôpital's Rule* gives us a means to evaluate such limts.

# L'Hôpital's Rule

**Observation:** Let $h(x) = \frac{f(x)}{g(x)}$ and

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x).$$

Assume that $f(x)$ and $g(x)$ have continuous derivatives with $g'(a) \neq 0$.
We know that for $x$ near $a$ we have that

$$\frac{f(x)}{g(x)} \cong \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

since $f(a) = 0 = g(a)$.
This might lead us to guess that if $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}. \qquad (*)$$

Moreover, since $f'(x)$ and $g'(x)$ are continuous with $g'(a) \neq 0$, we also have

$$\frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \qquad (**)$$

Combining $(*)$ and $(**)$ gives us

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

# L'Hospital's Rule

**Theorem: [ L'H$\hat{o}$pital's Rule]**

Assume that $f'(x)$ and $g'(x)$ exist near $x = a$, $g'(x) \neq 0$ near $x = a$ except possibly at $x = a$, and that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists (or is $\infty$ or $-\infty$ ).

Moreover, this rule remains valid for one-sided limits and for limits at $\pm\infty$.

**Note:** The proof of L'H$\hat{o}$pital's Rule uses an upgraded version of the MVT.

# L'H$\hat{o}$pital's Rule

**Example:** Evaluate

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

**Solution:** Let $f(x) = e^x - 1$ and $g(x) = x$. Then

$$\lim_{x \to 0} e^x - 1 = e^0 - 1 = 0 = \lim_{x \to 0} x \Rightarrow \text{ type } \frac{0}{0}.$$

Since $f'(x) = e^x$ and $g'(x) = 1$, by L'H$\hat{o}$pital's Rule

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1.$$

**Note:**

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

is the derivative of $f(x) = e^x$ at $x = 0$.

# L'Hôpital's Rule

**Example:** Evaluate

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}.$$

**Solution:** Let $f(x) = e^x - 1 - x$ and $g(x) = x^2$. Then

$$\lim_{x\to 0} e^x - 1 - x = 0 = \lim_{x\to 0} x^2 \Rightarrow \text{ type } \frac{0}{0}.$$

Then $f'(x) = e^x - 1$ and $g'(x) = 2x$ and

$$\lim_{x\to 0} e^x - 1 = 0 = \lim_{x\to 0} 2x \Rightarrow \lim_{x\to 0} \frac{f'(x)}{g'(x)} \text{ is type } \frac{0}{0}.$$

Let $F(x) = f'(x) = e^x - 1$ and $G(x) = g'(x) = 2x$, then $F'(x) = e^x$ and $G'(x) = 2$ so

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{F(x)}{G(x)} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

# L'H$\hat{o}$pital's Rule

**Example:** Evaluate

$$\lim_{x\to 0+} \frac{e^x - 1}{x^2}.$$

**Solution:** Let $f(x) = e^x - 1$ and $g(x) = x^2$. Then $f'(x) = e^x$ and $g'(x) = 2x$, so we have

$$\lim_{x\to 0+} \frac{e^x - 1}{x^2} = \lim_{x\to 0+} \frac{e^x}{2x}.$$

But

$$\lim_{x\to 0+} 2x = 0.$$

so by applying L'H$\hat{o}$pital's Rule again we get

$$\lim_{x\to 0+} \frac{e^x - 1}{x^2} = \lim_{x\to 0+} \frac{e^x}{2x} = \lim_{x\to 0+} \frac{e^x}{2} = \frac{1}{2}.$$

**Warning:** This is wrong since $\lim_{x\to 0+} e^x = 1 \neq 0$ so

$$\lim_{x\to 0+} \frac{e^x - 1}{x^2} = \lim_{x\to 0+} \frac{e^x}{2x} = \infty.$$

# L'Hôpital's Rule

**Example:** Evaluate

$$\lim_{x \to \infty} \frac{\ln(x)}{x}.$$

**Solution:** Let $f(x) = \ln(x)$ and $g(x) = x$. Then

$$\lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x),$$

so this is an indeterminate form of the type $\frac{\infty}{\infty}$.

Differentiating $f(x)$ and $g(x)$ gives us $f'(x) = \frac{1}{x}$ and $g'(x) = 1$. Therefore,

$$\begin{aligned} \lim_{x \to \infty} \frac{f'(x)}{g'(x)} &= \lim_{x \to \infty} \frac{\frac{1}{x}}{1} \\ &= \lim_{x \to \infty} \frac{1}{x} \\ &= 0. \end{aligned}$$

Thus

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = 0.$$

# L'H$\hat{o}$pital's Rule

**Remark:** Up until now we have dealt with two types of indeterminate forms which we have denoted by $\frac{0}{0}$ and $\frac{\infty}{\infty}$. There are five more standard indeterminate forms which we will denote by

$$0 \cdot \infty, \quad \infty - \infty, \quad 1^{\infty}, \quad \infty^{0}, \quad \text{and} \quad 0^{0}.$$

For example, an indeterminate form of type $0 \cdot \infty$ arises from the function $h(x) = f(x)g(x)$ when

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = \infty.$$

Similarly, the function $(g(x))^{f(x)}$ would produce an indeterminate form of type $\infty^{0}$.

**Note:** All of the above forms can be *converted* to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

# L'H$\hat{o}$pital's Rule

**Example:** Evaluate

$$\lim_{x\to 0+} x\ln(x).$$

**Solution:** This is an indeterminate form of type $0\cdot\infty$ since

$$\lim_{x\to 0^+} x = 0$$

and

$$\lim_{x\to 0^+} \ln(x) = -\infty.$$

We can rewrite this example as

$$\lim_{x\to 0+} \frac{\ln(x)}{\frac{1}{x}}$$

which is type $\frac{\infty}{\infty}$. L'H$\hat{o}$pital's Rule gives us that

$$\begin{aligned}\lim_{x\to 0^+} x\ln(x) &= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}\\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(\frac{-1}{x^2}\right)}\\ &= 0.\end{aligned}$$

# L'H$\hat{o}$pital's Rule

**Example:** Evaluate

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

**Solution:** This is type $1^\infty$. We write

$$\left(1+\frac{1}{x}\right)^x = e^{\ln\left(\left(1+\frac{1}{x}\right)^x\right)} = e^{x\ln\left(1+\frac{1}{x}\right)}.$$

Since $x\ln\left(1+\frac{1}{x}\right)$ is type $0\cdot\infty$, we rewrite this as

$$\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$$

which is type $\frac{0}{0}$. L'H$\hat{o}$pital's Rule gives us that

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{\frac{\left(\frac{-1}{x^2}\right)}{1+\frac{1}{x}}}{\left(\frac{-1}{x^2}\right)}\\ &= \frac{1}{1+\frac{1}{x}}\\ &= 1.\end{aligned}$$

So

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^x\right)} = e.$$

# L'H$\hat{o}$pital's Rule



**Problem:**

Show that

$$\lim_{x \to 0} \frac{4(e^{x^3} - 1 - x^3 - \frac{x^6}{2})^2}{x^6 \tan(x^7) \sin(2x^5)} = \frac{1}{18}.$$