L’H$\hat{o}$pital’s Rule: Part II

Created by

Barbara Forrest and Brian Forrest

COPYRIGHTED

L'Hospital's Rule

Theorem: [L'H$\hat{o}$pital's Rule]

Assume that $f'(x)$ and $g'(x)$ exist near $x = a$, $g'(x) \neq 0$ near $x = a$ except possibly at $x = a$, and that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists (or is ∞ or $-\infty$).

Moreover, this rule remains valid for one-sided limits and for limits at $\pm\infty$.

COPYRIGHTED

L'Hôpital's Rule

Remark: Up until now we have dealt with two types of indeterminate forms which we have denoted by $\frac{0}{0}$ and $\frac{\infty}{\infty}$. There are five more standard indeterminate forms which we will denote by

$$0 \cdot \infty, \quad \infty - \infty, \quad 1^{\infty}, \quad \infty^{0}, \quad \text{and} \quad 0^{0}.$$

For example, an indeterminate form of type $0 \cdot \infty$ arises from the function $h(x) = f(x)g(x)$ when

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = \infty.$$

Similarly, the function $(g(x))^{f(x)}$ would produce an indeterminate form of type ∞^{0}.

Note: All of the above forms can be *converted* to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

COPYRIGHTED

L'Hôpital's Rule

Example: Evaluate

$$\lim_{x \to 0+} x \ln(x).$$

Solution: This is an indeterminate form of type $0 \cdot \infty$ since

$$\lim_{x \to 0^+} x = 0$$

and

$$\lim_{x \to 0^+} \ln(x) = -\infty.$$

We can rewrite this example as

$$\lim_{x \to 0+} \frac{\ln(x)}{\frac{1}{x}}$$

which is type $\frac{\infty}{\infty}$. L'Hôpital's Rule gives us that

$$\begin{aligned} \lim_{x \to 0^+} x \ln(x) &= \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \\ &= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\left(\frac{-1}{x^2}\right)} \\ &= 0. \end{aligned}$$

COPYRIGHTED

L'Hôpital's Rule

Example: Evaluate

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.$$

Solution: This is type 1^{∞}. We write

$$\left(1+\frac{1}{x}\right)^{x} = e^{\ln\left(\left(1+\frac{1}{x}\right)^{x}\right)} = e^{x\ln\left(1+\frac{1}{x}\right)}.$$

Since $x\ln\left(1+\frac{1}{x}\right)$ is type $0\cdot\infty$, we rewrite this as

$$\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$$

which is type $\frac{0}{0}$. L'Hôpital's Rule gives us that

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{\frac{\left(\frac{-1}{x^2}\right)}{1+\frac{1}{x}}}{\left(\frac{-1}{x^2}\right)}\\ &= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\\ &= 1.\end{aligned}$$

So

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e^{\lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^{x}\right)} = e.$$

COPYRIGHTED

L'H$\hat{o}$pital's Rule

Problem:

Show that

$$\lim_{x\to 0} \frac{4(e^{x^3} - 1 - x^3 - \frac{x^6}{2})^2}{x^6 \tan(x^7) \sin(2x^5)} = \frac{1}{18}.$$

COPYRIGHTED