Applications of the MVT: Increasing Function Theorem

Created by

Barbara Forrest and Brian Forrest

Functions

Definition: [Increasing and Decreasing Functions]

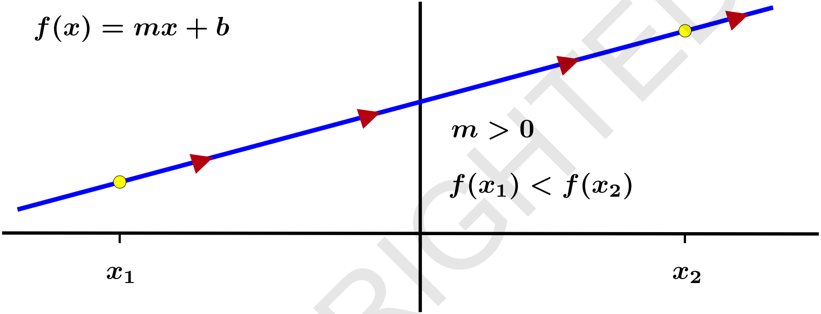
Suppose that $f(x)$ is defined on an interval I.

i) We say that $f(x)$ is *increasing on* I if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$.
ii) We say that $f(x)$ is *decreasing on* I if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$.
iii) We say that $f(x)$ is *non-decreasing on* I if $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$.
iv) We say that $f(x)$ is *non-increasing on* I if $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$.

Such functions are said to be *monotonic* on I.

Question: How can we determine if a function $f(x)$ is either increasing or decreasing on an interval I?

Increasing Function Theorem



Observation: Assume that

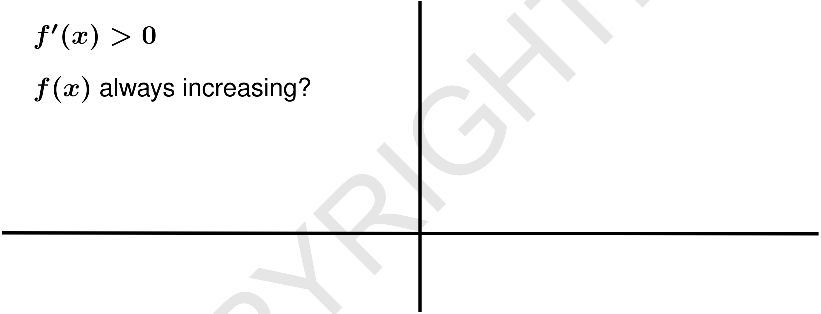
$$f(x) = mx + b.$$

If $m > 0$, the graph of the function slopes upward as we move from left to right. In other words, if $x_1 < x_2$, then

$$f(x_1) = mx_1 + b < mx_2 + b = f(x_2).$$

Note: $f'(x) = m > 0$ for all $x \in \mathbb{R}$.

Increasing Function Theorem



Question: If $f(x)$ is such that $f'(x) > 0$ for all $x \in I$, is $f(x)$ increasing on I?

Increasing Function Theorem

Theorem: [The Increasing/Decreasing Function Theorem]

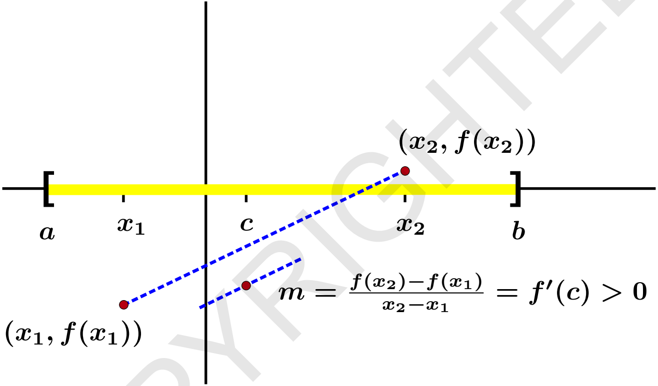
i) Let I be an interval and assume that $f'(x) > 0$ for all $x \in I$. If $x_1 < x_2$ are two points in I, then
$$f(x_1) < f(x_2).$$
That is, $f(x)$ is increasing on I.

ii) Let I be an interval and assume that $f'(x) \geq 0$ for all $x \in I$. If $x_1 < x_2$ are two points in I, then
$$f(x_1) \leq f(x_2).$$
That is, $f(x)$ is non-decreasing on I.

iii) Let I be an interval and assume that $f'(x) < 0$ for all $x \in I$. If $x_1 < x_2$ are two points in I, then
$$f(x_1) > f(x_2).$$
That is, $f(x)$ is decreasing on I.

iv) Let I be an interval and assume that $f'(x) \leq 0$ for all $x \in I$. If $x_1 < x_2$ are two points in I, then
$$f(x_1) \geq f(x_2).$$
That is, $f(x)$ is non-increasing on I.

Increasing Function Theorem



i) Let I be an interval and assume that $f'(x) > 0$ for all $x \in I$. If $x_1 < x_2$ are two points in I, then

$$f(x_1) < f(x_2).$$

Increasing Function Theorem

Proof of i): Assume that $f'(x) > 0$ for all $x \in I$. Let $x_1, x_2 \in I$ with $x_1 < x_2$. By the MVT there exists $c \in (x_1, x_2)$ with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0.$$

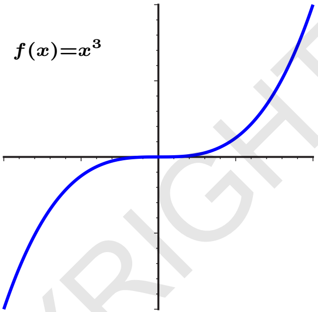
Since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) > 0$$

and hence that

$$f(x_2) > f(x_1).$$

Increasing Function Theorem



Question: If $f(x)$ is increasing on an interval I and differentiable on I, then must $f'(x) > 0$ for all $x \in I$?

Solution: Let $f(x) = x^3$. Since $f'(x) = 3x^2$, we have

$$f'(0) = 0$$

but $f(x)$ is increasing on all of $\mathbb{R}$.

Increasing Function Theorem

Question:

1) Is the function $f(x) = x^3$ increasing on $[0, 1]$? **Yes!**
2) If $f(x)$ is everywhere differentiable and if $f'(c) > 0$, does this mean that there is an open interval (a, b) containing c on which $f(x)$ is increasing? **No!**