Applications of the MVT: The First Derivative Test

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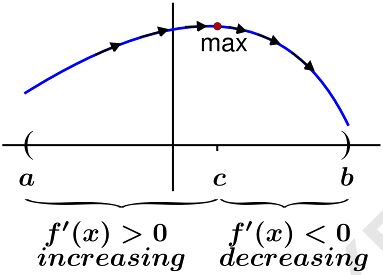
First Derivative Test

Definition: [Critical Point]

A point c in the domain of a function $f(x)$ is called a *critical point* if either $f'(c) = 0$ or $f'(c)$ does not exist.

Problem: Given a critical point c for $f(x)$, how can we determine if it is either a local maximum or a local minimum?

First Derivative Test



Theorem: [First Derivative Test]

Assume that c is a critical point of $f(x)$, and $f(x)$ is continuous at c.

i) If there is an interval (a, b) containing c such that

$$f'(x) < 0 \quad \text{for all } x \in (a, c)$$
$$\text{and} \quad f'(x) > 0 \quad \text{for all } x \in (c, b),$$

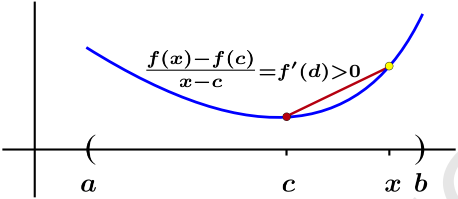
then $f(x)$ has a local minimum at $x = c$.

ii) If there is an interval (a, b) containing c such that

$$f'(x) > 0 \quad \text{for all } x \in (a, c)$$
$$\text{and} \quad f'(x) < 0 \quad \text{for all } x \in (c, b),$$

then $f(x)$ has a local maximum at $x = c$.

First Derivative Test



Proof of i)

Assume that $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$.

i) Let $x \in (a, c)$. Then

$$\frac{f(x) - f(c)}{x - c} = f'(d) < 0 \Rightarrow f(x) - f(c) > 0.$$

ii) Let $x \in (c, b)$. Then

$$\frac{f(x) - f(c)}{x - c} = f'(d) > 0 \Rightarrow f(x) - f(c) > 0.$$

First Derivative Test

Example: Find all the critical points for

$$f(x) = \frac{x^3}{3} - x$$

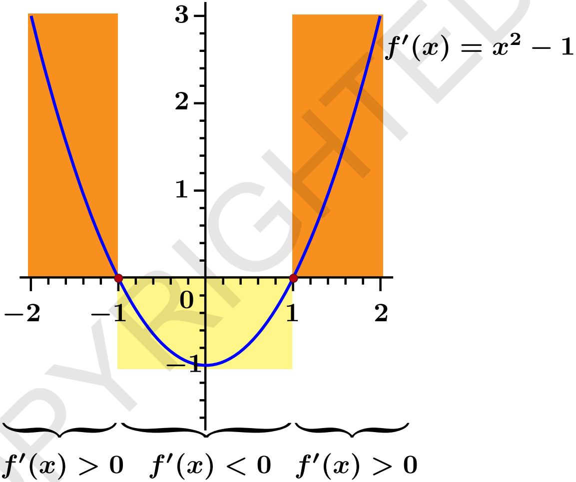
and determine if each is a local maximum, a local minimum, or neither.

Solution: We need to examine the first derivative:

$$f'(x) = x^2 - 1 = (x-1)(x+1)$$

so the only critical points are $x = -1$ and $x = 1$.

First Derivative Test



1) $f'(x)$ changes from positive to negative at $x = -1$ $\Rightarrow$ local maximum.
2) $f'(x)$ changes from negative to positive at $x = 1$ $\Rightarrow$ local minimum.

First Derivative Test