Curve Sketching
Part II

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Curve Sketching

Recall:

Previously we used information about limits and continuity to sketch the graphs of various functions.

Strategy: Basic Curve Sketching (Part I)

Step 1: Determine the domain of $f(x)$.

Step 2: Determine any symmetries that the graph may have. In particular, test to see if the function is either even or odd.

Step 3: Determine, if possible, where the function changes sign and plot these points.

Step 4: Find any discontinuity points for $f(x)$.

Step 5: Evaluate the relevant one-sided and two-sided limits at the points of discontinuity and identify the nature of the discontinuities. In particular, indicate any removable discontinuity with a small circle to denote the hole.

Step 6: Using the information from step (5), draw any vertical asymptotes.

Step 7: Find any horizontal asymptotes by evaluating the limits of the function at $\pm\infty$, if applicable. Draw the horizontal asymptotes on your plot.

Step 8: Finally, use the information you have gathered above to construct as accurate a sketch as possible for the graph of the given function. It is often helpful to plot a few sample points as a guide.

Example: Basic Curve Sketching (Part I)

Example (revisited): Sketch the graph of

$$f(x) = \frac{xe^x}{x^3 - x}\,.$$

Solution:

Step 1: This function is defined everywhere except when the denominator is zero:

$$x^3 - x = x(x-1)(x+1) = 0.$$

That is, everywhere except when $x = 0$ and $x = \pm 1$.

Step 2: Since

$$f(-x) = \frac{-xe^{-x}}{-x^3 + x} \neq f(x) \text{ and } f(-x) = \frac{xe^{-x}}{x^3 - x} \neq -f(x)$$

$f(x)$ is neither even nor odd. In fact, there are no obvious symmetries.

Example: Basic Curve Sketching (Part I)

Solution (continued):

Step 3: We first observe that

$$f(x) = \frac{xe^x}{x^3 - x} = \frac{e^x}{x^2 - 1}$$

for all $x \neq 0$. Since e^x is never 0, the function is never 0.

The IVT tells us that we could only have a sign change at a point of discontinuity.

Example: Basic Curve Sketching (Part I)

Solution (continued):

Step 4:

$$f(x) = \frac{xe^x}{x^3 - x}$$

The function is the ratio of two continuous functions. Therefore, $f(x)$ is discontinuous only at $x = 0$, $x = -1$ and $x = 1$ since these are the only points where the denominator is 0.

Example: Basic Curve Sketching (Part I)

Solution (continued):

Step 5:

1) e^x is always positive and since $f(x) = \frac{xe^x}{x^3-x} = \frac{e^x}{x^2-1}$ for all $x \neq 0$, there is no sign change at $x = 0$.

2) $f(x)$ goes from positive to negative as we move across $x = -1$ and then from negative to positive as we cross $x = 1$ (moving left to right).

Example: Basic Curve Sketching (Part I)

Solution (continued):

Step 5 (continued):

3) Since for $x \neq 0$, we have

$$f(x) = \frac{xe^x}{x^3 - x} = \frac{e^x}{x^2 - 1}$$

this gives us that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x}{x^2 - 1} = -1.$$

So $x = 0$ is a removable discontinuity.

Example: Basic Curve Sketching (Part I)

Solution (continued):

Step 5 (continued):

4) $x = 1$ and $x = -1$ are both vertical asymptotes for $f(x)$.
5) Since $f(x) > 0$ if $x > 1$ or $x < -1$, and $f(x) < 0$ if $x \neq 0$ and $-1 < x < 1$, we get that

$$\lim_{x \to 1^+} f(x) = \infty,$$

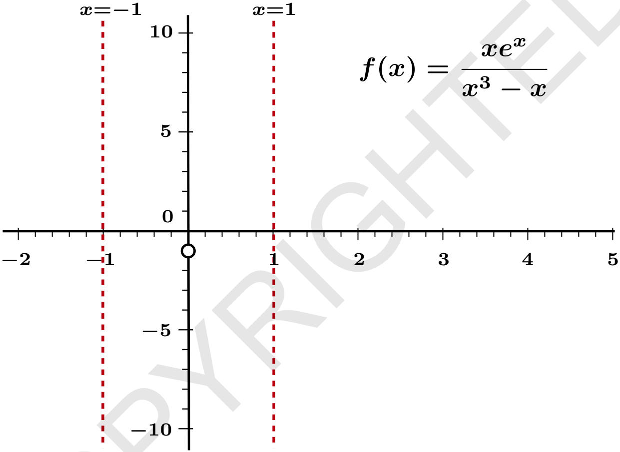
$$\lim_{x \to 1^-} f(x) = -\infty,$$

and

$$\lim_{x \to -1^+} f(x) = -\infty,$$

$$\lim_{x \to -1^-} f(x) = \infty.$$

Example: Basic Curve Sketching (Part I)



Solution (continued):

Step 6: Draw the vertical asymptotes and indicate the removable discontinuities on the plot.

Example: Basic Curve Sketching (Part I)

Solution (continued):

Step 7:

$$f(x) = \frac{xe^x}{x^3 - x}$$

1) Since e^x grows much more rapidly than any polynomial for large positive values of x, we have

$$\lim_{x \to \infty} f(x) = \infty.$$

2) But since e^x becomes very small for large negative values of x, we have

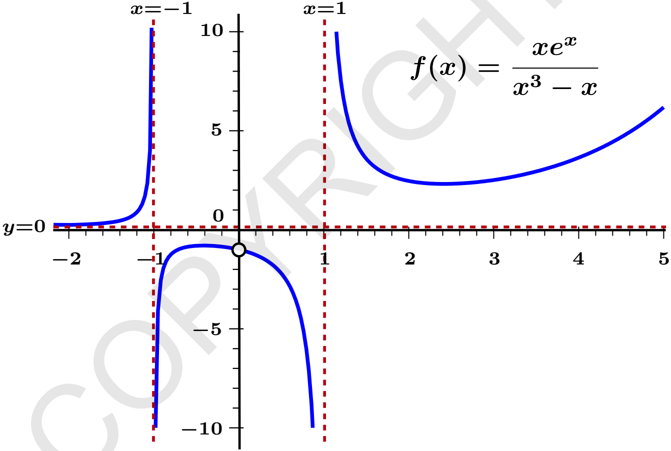
$$\lim_{x \to -\infty} f(x) = 0.$$

Thus, $y = 0$ is a horizontal asymptote as $x \to -\infty$.

Example: Basic Curve Sketching (Part I)

Solution (continued):

Step 8: Sketch the graph of $f(x)$ using all of this information.



Basic Curve Sketching (Part II)

We can now use the information obtained from the derivative to refine the sketch of the function.

Strategy: Basic Curve Sketching (Part II)

Step 1: Complete the steps for *Curve Sketching: Part 1*.

Step 2: Calculate $f'(x)$.

Step 3: Identify any critical points: locate where $f'(x) = 0$ or where $f'(x)$ does not exist.

Step 4: Determine where $f(x)$ is increasing or decreasing by analysing the sign of $f'(x)$ between the critical points.

Step 5: Test the critical points to determine if they are local maxima, local minima, or neither.

Step 6: Find $f''(x)$.

Step 7: Locate where $f''(x) = 0$ or where $f''(x)$ does not exist. Use these points to divide the Real number line into intervals. Determine the concavity of $f(x)$ by analysing the sign of $f''(x)$ inside these intervals (if possible).

Step 8: Identify any points of inflection.

Step 9: Incorporate this information into your original sketch.

Example: Basic Curve Sketching (Part II)

Example (revisited): Enhance the sketch the graph of

$$f(x) = \frac{xe^x}{x^3 - x}$$

by using the Basic Curve Sketching Steps (Part II).

Solution:

Step 1: Calculate $f'(x)$. Taking the derivative of $f(x)$ we get

$$f'(x) = \frac{(x^2 - 2x - 1)e^x}{(x^2 - 1)^2}$$

for $x \neq 0$, $x \neq \pm 1$.

Example: Basic Curve Sketching (Part II)

Solution (continued):

$$f(x) = \frac{xe^x}{x^3 - x}$$

Step 2: A critical point $x = c$ occurs when c is in the domain of $f(x)$ and either $f'(c)$ does not exist or $f'(c) = 0$. Now since $e^x > 0$ for all x and the denominator is always positive for all $x \neq \pm 1$, it follows that $f'(x) = 0$ when

$$x^2 - 2x - 1 = 0.$$

The quadratic formula tells us that these critical points occur when

$$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

Moreover,

$$1 + \sqrt{2} \approx 2.414213562$$

and

$$1 - \sqrt{2} \approx -0.414213562.$$

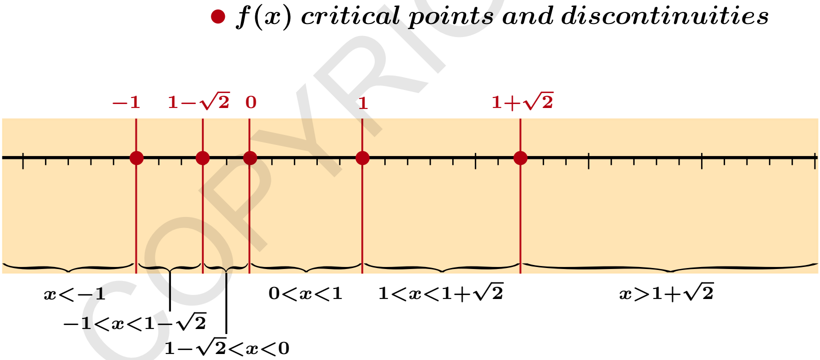
This suggests that the local minimum is located at $x \approx 2.4141213562$ and the local maximum is located at $x \approx -0.414213562$. We will confirm these results in the next few steps.

Example: Basic Curve Sketching (Part II)

Solution (continued):

$$f(x) = \frac{xe^x}{x^3 - x}$$

Step 3: We can determine where $f(x)$ is increasing or decreasing by analysing the sign of $f'(x)$ in particular intervals and then applying the Increasing/Decreasing Function Theorem.



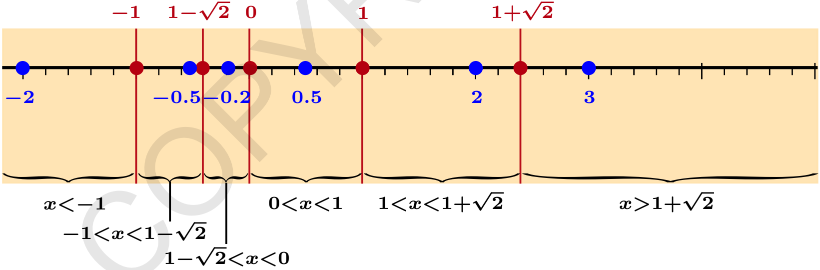
Example: Basic Curve Sketching (Part II)

Solution (continued):

$$f(x) = \frac{xe^x}{x^3 - x}$$

Step 3: We can determine where $f(x)$ is increasing or decreasing by analysing the sign of $f'(x)$ in particular intervals and then applying the Increasing/Decreasing Function Theorem.

- $f(x)$ *critical points and discontinuities*
- $f'(x)$ *test points*

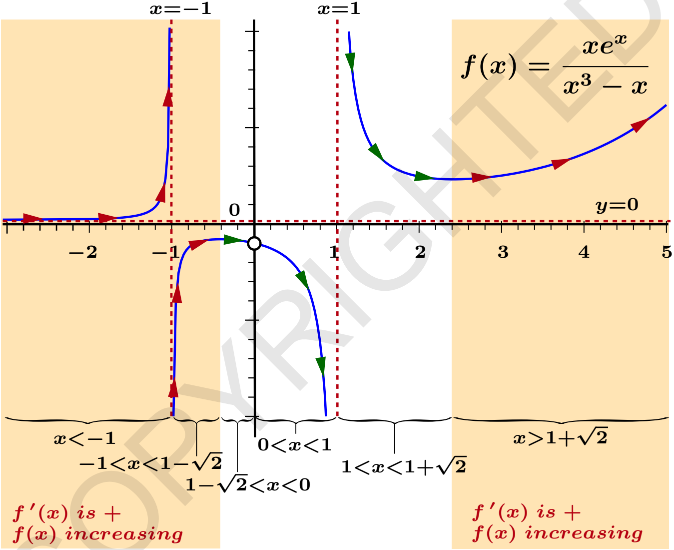


Example: Basic Curve Sketching (Part II)

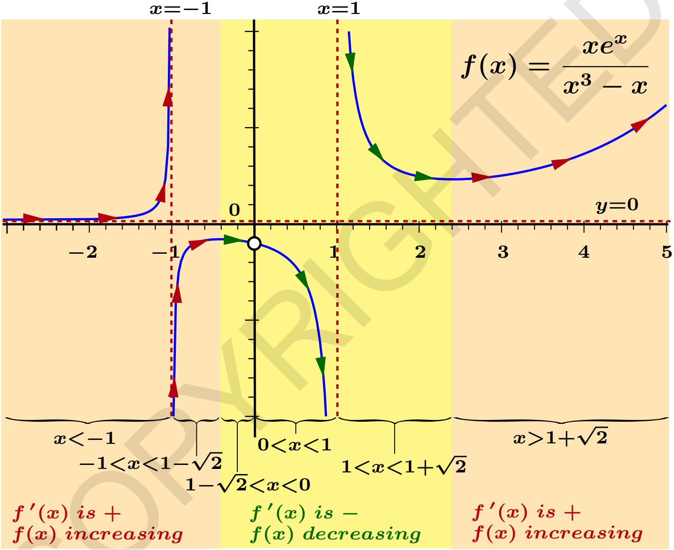
Step 3 (continued): Apply the Increasing/Decreasing Function Theorem. You should verify these calculations.

Interval	Test Point	Calculate $x^2 - 2x - 1$	$f'(x)$ positive or negative	$f(x)$ increasing or decreasing
$x < -1$	$x = -2$	$(-2)^2 - 2(-2) - 1 = +7$	$f'(x) > 0$ (positive)	$f(x)$ increasing
$-1 < x < 1 - \sqrt{2}$	$x = -0.5$	$(-0.5)^2 - 2(-0.5) - 1 = +0.25$	$f'(x) > 0$ (positive)	$f(x)$ increasing
$1 - \sqrt{2} < x < 0$	$x = -0.2$	$(0.2)^2 - 2(0.2) - 1 = -1.36$	$f'(x) < 0$ (negative)	$f(x)$ decreasing
$0 < x < 1$	$x = 0.5$	$(0.5)^2 - 2(0.5) - 1 = -1.75$	$f'(x) < 0$ (negative)	$f(x)$ decreasing
$1 < x < 1 + \sqrt{2}$	$x = 2$	$(2)^2 - 2(2) - 1 = -1$	$f'(x) < 0$ (negative)	$f(x)$ decreasing
$x > 1 + \sqrt{2}$	$x = 3$	$(3)^2 - 2(3) - 1 = +2$	$f'(x) > 0$ (positive)	$f(x)$ increasing

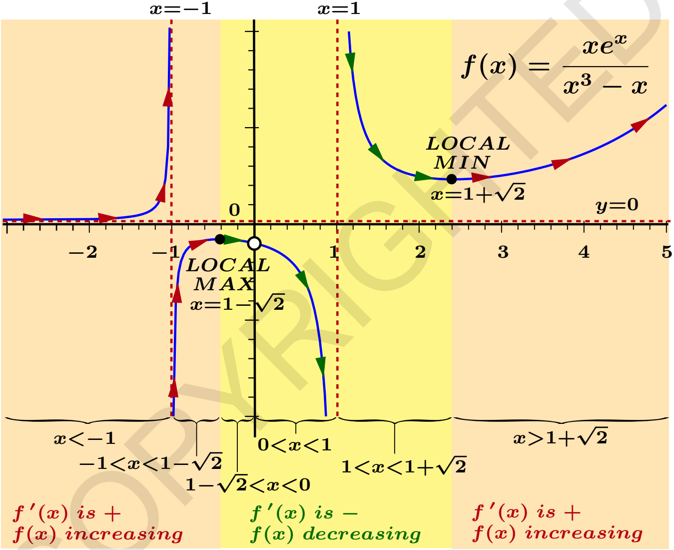
Example: Basic Curve Sketching (Part II)



Example: Basic Curve Sketching (Part II)



Example: Basic Curve Sketching (Part II)



Example: Basic Curve Sketching (Part II)