Applications of the MVT: Concavity

Created by

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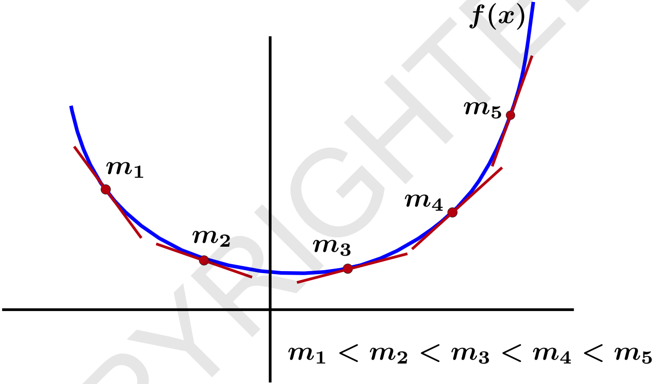
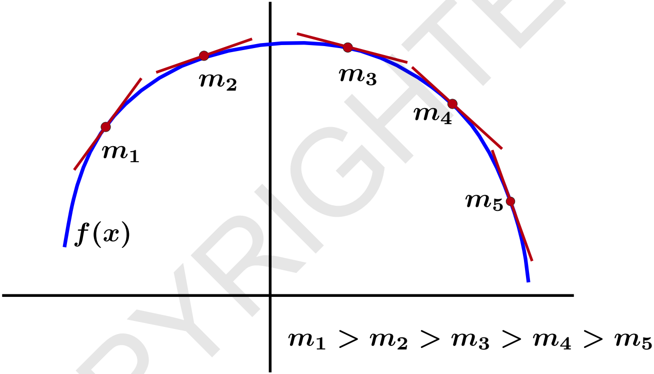
Concavity



Question: If $f''(x) > 0$ on some interval I, what can we say about the shape of the graph of the function $f(x)$?

Observation: $f'(x)$ is increasing so the tangent lines to $f(x)$ rotate counter-clockwise $\Rightarrow$ $f(x)$ is **concave upwards**.

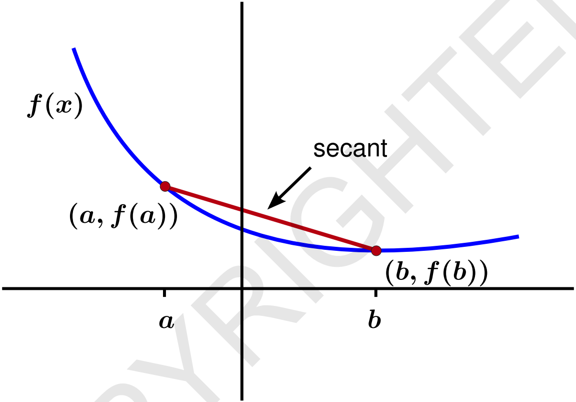
Concavity



Question: What if $f''(x) < 0$ on I?

Observation: $f'(x)$ is decreasing so tangent lines to $f(x)$ rotate clockwise $\Rightarrow$ $f(x)$ is **concave downwards.**

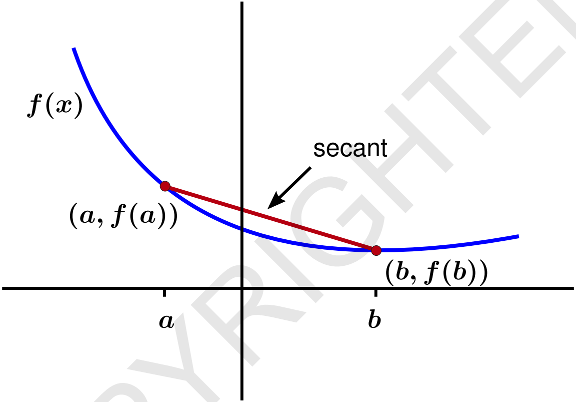
Concave Upwards



Question: How can we quantify *concave upwards*?

Observation: The secant line lies **above** the graph of $f(x)$.

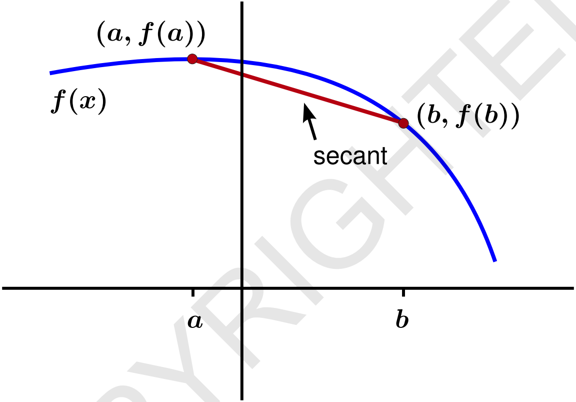
Concave Upwards



Definition: [Concave Upwards]

The graph of $f(x)$ is *concave upwards* on an interval I if for every pair of points a and b in I, the secant line joining $(a, f(a))$ and $(b, f(b))$ lies *above* the graph of $f(x)$.

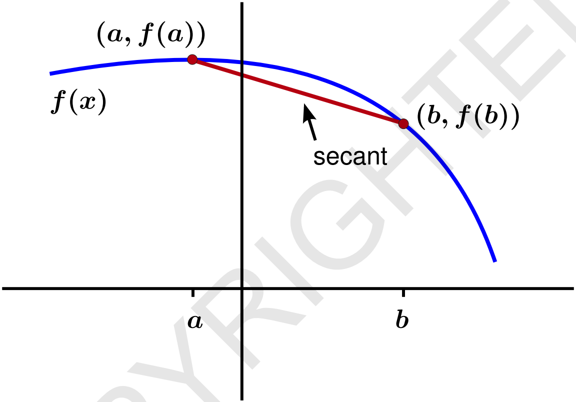
Concave Downwards



Question: How can we quantify *concave downwards*?

Observation: The secant line lies **below** the graph of $f(x)$.

Concave Downwards



Definition: [Concave Downwards]

The graph of $f(x)$ is *concave downwards* on an interval I if for every pair of points a and b in I, the secant line joining $(a, f(a))$ and $(b, f(b))$ lies *below* the graph of $f(x)$.

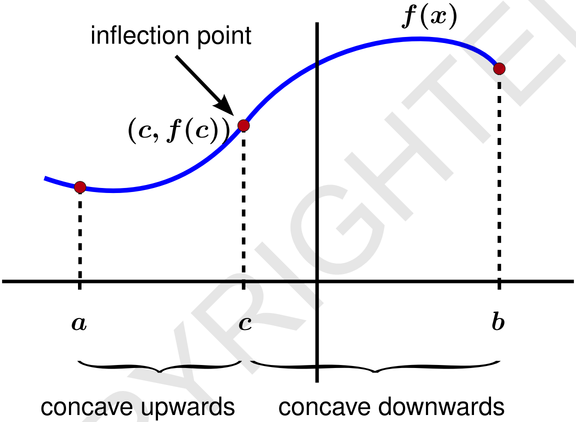
Second Derivative Test for Concavity

Theorem: [Second Derivative Test for Concavity]

i) If $f''(x) > 0$ for each x in an interval I, then the graph of $f(x)$ is concave upwards on I.

ii) If $f''(x) < 0$ for each x in an interval I, then the graph of $f(x)$ is concave downwards on I.

Inflection Points



Example:

1) $f(x)$ is concave upwards on $[a, c]$.
2) $f(x)$ is concave downwards on $[c, b]$.
3) $f(x)$ *changes its concavity* at $x = c$. The point $(c, f(c))$ is called an *inflection point*.

Inflection Points

Definition: [Inflection Point]

A point c is called an *inflection point* for the function $f(x)$ if

i) $f(x)$ is continuous at $x = c$, and
ii) the concavity of $f(x)$ changes at $x = c$.

Important Note:

1) Typically an inflection point $x = c$ would occur when the second derivative changes from positive to negative, or vice versa.
2) If $f''(x)$ is continuous, the Intermediate Value Theorem requires that $f''(c) = 0$.

Inflection Points

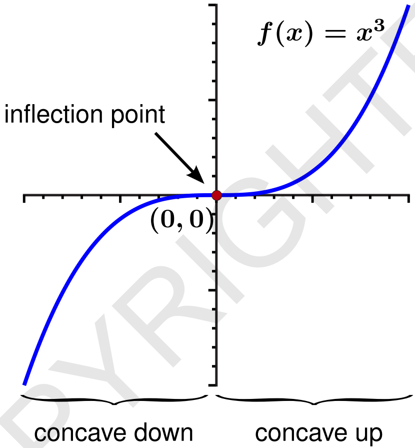
Theorem: [Test for Inflection Points]

If $f''(x)$ is continuous at $x = c$ and $(c, f(c))$ is an inflection point for $f(x)$, then

$$f''(c) = 0.$$

WARNING: This theorem shows us how to *locate candidates* for inflection points. However, $f''(c) = 0$ does *not* mean that an inflection point always occurs when $x = c$.

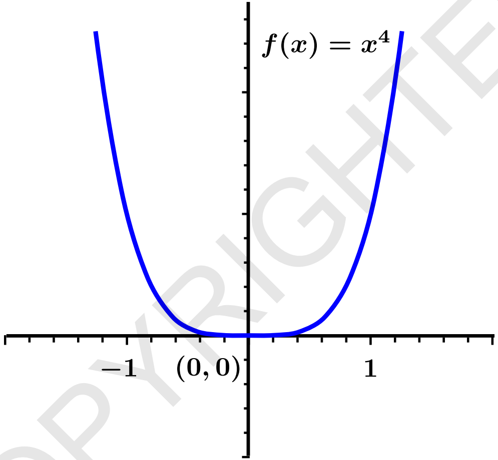
Inflection Points



Example: Let $f(x) = x^3$. Then $f''(x) = 6x$.

1) $f''(x) < 0$ if $x < 0$ so $f(x)$ is concave downwards on $(-\infty, 0]$.
2) $f''(x) > 0$ if $x > 0$ so $f(x)$ is concave upwards on $[0, \infty)$.
3) The function has a point of inflection at $x = 0$.

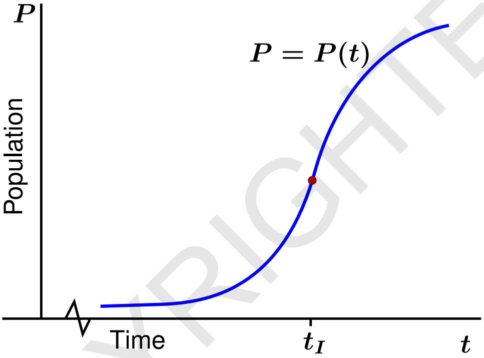
Inflection Points



Example: Let $f(x) = x^4$. Then $f''(x) = 12x^2$.

1) $f''(x) > 0$ for all $x \neq 0$ so $f(x)$ is concave upwards on $\mathbb{R}$.
2) $f''(0) = 0$ but the function does *not* have a point of inflection at $x = 0$.

Inflection Points



Example:

The diagram represents the graph of the population P of a particular bacteria over time t in a restricted environment.

Important Fact: The point of inflection occurs at the time t_I when the growth rate of the bacteria is at its highest value.