Applications of the MVT: Comparing Functions through their Derivatives

Created by

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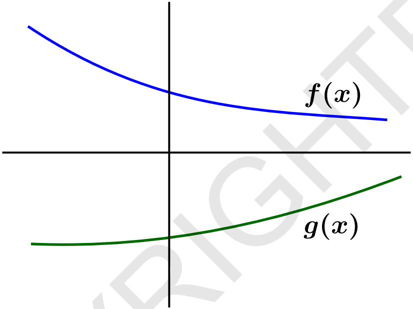
Comparing Functions



Remark:

If $f(x)$ and $g(x)$ are such $f'(x) \leq g'(x)$ on an interval I, we **cannot** conclude that $f(x) \leq g(x)$ on I.

However, if we assume that

$$f(a) = g(a)$$

we can say more.

Comparing Functions

Theorem:

Assume that $f(x)$ and $g(x)$ are continuous at $x = a$ with $f(a) = g(a)$.

i) If both $f(x)$ and $g(x)$ are differentiable for $x > a$ and if $f'(x) \leq g'(x)$ for all $x > a$, then

$$f(x) \leq g(x)$$

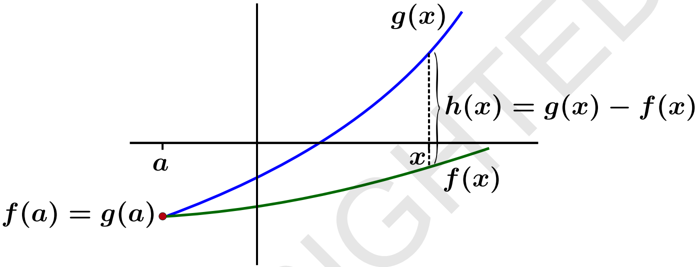
for all $x > a$.

ii) If both $f(x)$ and $g(x)$ are differentiable for $x < a$ and if $f'(x) \leq g'(x)$ for all $x < a$, then

$$f(x) \geq g(x)$$

for all $x < a$.

Comparing Functions



Proof of i): Assume that $f(x)$ and $g(x)$ are differentiable for $x > a$, $f'(x) \leq g'(x)$ for all $x > a$, and that $f(a) = g(a)$.

Let

$$h(x) = g(x) - f(x).$$

Then

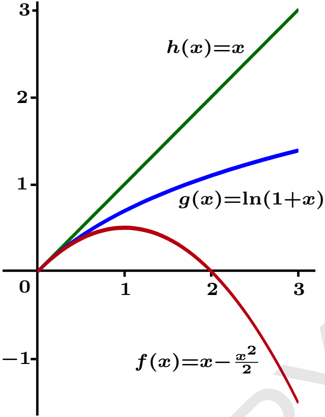
$$h(a) = g(a) - f(a) = 0 \quad \text{and} \quad h'(x) = g'(x) - f'(x) \geq 0$$

for all $x > a$. So $h(x)$ is non-decreasing on $[a, \infty)$ and

$$h(x) = g(x) - f(x) \geq 0$$

for all $x > a$.

Comparing Functions



Example: Show that

$$x - \frac{x^2}{2} < \ln(x+1) < x$$

for all $x > 0$.

Solution:

Note that if $f(x) = x - \frac{x^2}{2}$, $g(x) = \ln(x+1)$ and $h(x) = x$, then $f(0) = g(0) = h(0) = 0$.

We also have that

$$f'(x) = 1 - x, \; g'(x) = \frac{1}{x+1}, \; h'(x) = 1.$$

For $x > 0$

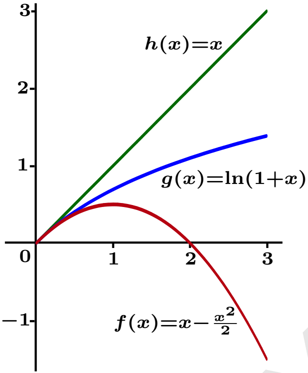
$$g'(x) = \frac{1}{x+1} < h'(x) = 1 \Rightarrow g(x) < h(x)$$

and

$$1 - x^2 = (1-x)(1+x) < 1 \quad \Rightarrow \quad f'(x) = 1 - x < g'(x) = \frac{1}{x+1}$$

$$\Rightarrow \quad f(x) < g(x).$$

$\lim_{n\to\infty}(1+\frac{1}{n})^n$



Example: Show that

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e.$$

Solution: Since

$$x-\frac{x^2}{2} < \ln(1+x) < x$$

for all $x > 0$, we get

$$1-\frac{x}{2} < \frac{\ln(1+x)}{x} < 1$$

for all $x > 0$. Since

$$\lim_{x\to 0+} 1-\frac{x}{2} = 1 = \lim_{x\to 0+} 1, \text{ we have } \lim_{x\to 0+}\frac{\ln(1+x)}{x} = 1.$$

As $\frac{1}{n} \to 0$ when $n \to \infty$,

$$\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n}))}{\frac{1}{n}} = \lim_{n\to\infty}\ln(1+\frac{1}{n})^n = 1.$$

So

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty} e^{\ln(1+\frac{1}{n})^n} = e^{(\lim_{n\to\infty}\ln(1+\frac{1}{n})^n)} = e.$$

$\lim_{n\to\infty}(1+\frac{\alpha}{n})^n$

Theorem:

Let $\alpha \in \mathbb{R}$. Then

$$\lim_{n\to\infty}(1+\frac{\alpha}{n})^n = e^{\alpha}.$$