Applications of the MVT: Functions with Bounded Derivatives

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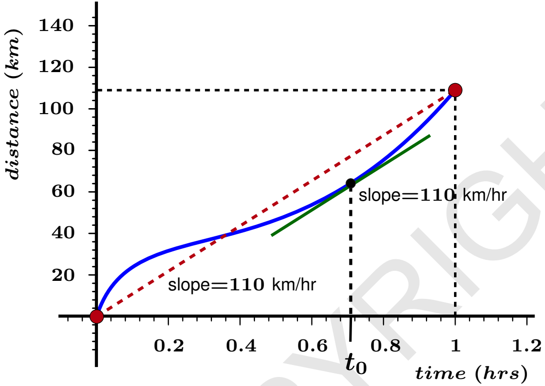
A Problem



Problem 1: A car travels forward a distance of 110 km in one hour along a road with a posted speed limit of 100 km/hr. Prove that at some point in the journey the car was speeding.

Average velocity $= 110$ km/hr.

At t_0 the tangent line is parallel to the secant line $\Rightarrow v(t_o) = 110$ km/hr.

Problem 2: If a car travels at most 100 km/hr, what is the maximum distance it could travel in exactly 1 hour?

A Problem

Observation: Assume that $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b) with

$$m \leq f'(x) \leq M$$

for each $x \in (a, b)$.

Let $x \in [a, b]$. Then Mean Value Theorem is true on the interval $[a, x]$. Hence there exists a c between a and x such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

Since $m \leq f'(x) \leq M$, we get that

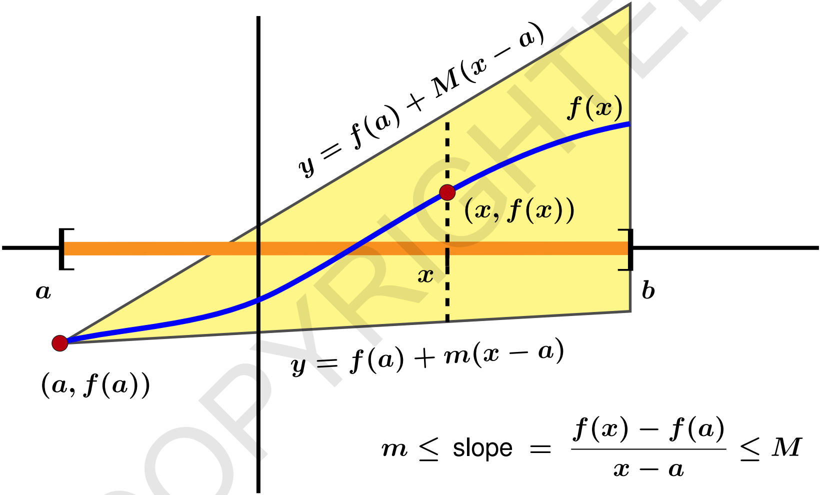
$$m \leq \frac{f(x) - f(a)}{x - a} \leq M.$$

So

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

for all $x \in [a, b]$.

MVT and Bounded Derivatives



Assume that $m \leq f'(x) \leq M$ for all $x \in [a, b]$.

Functions with Bounded Derivatives

Theorem: [Bounded Derivative Theorem]

Assume that $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and that

$$m \leq f'(x) \leq M$$

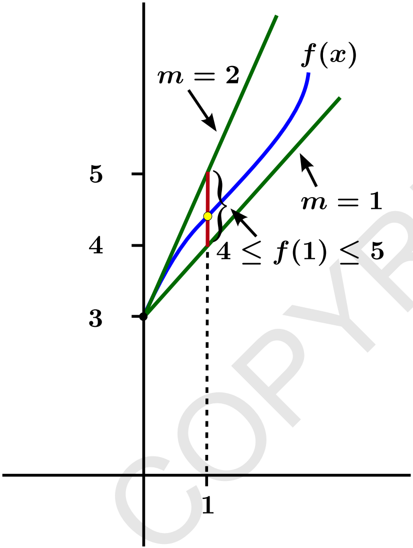
for each $x \in (a, b)$. Then

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

for all $x \in [a, b]$.

Remark: If we return to the scenario of a car traveling one hour along a road without exceeding a speed of 100 km/hr, then the previous theorem tells us immediately that the maximum distance the car could have traveled in that time frame was 100 km as we expected.

Functions with Bounded Derivatives



Example: Assume that

$$f(0) = 3$$

and that

$$1 \le f'(x) \le 2$$

for all $x \in [0, 1]$. Show that

$$4 \le f(1) \le 5.$$

Solution: We know that

$$f(0)+1\cdot(1-0) \le f(1) \le f(0)+2\cdot(1-0)$$

so

$$4 = 3 + 1 \le f(1) \le 3 + 2 = 5.$$