Applications of the MVT: Antiderivatives

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Antiderivatives

Problem: Given a function $f(x)$ does there exist a function $F(x)$ so that

$$F'(x) = f(x)?$$

Definition: [Antiderivative]

Given a function $f(x)$, an *antiderivative* is a function $F(x)$ such that

$$F'(x) = f(x).$$

If $F'(x) = f(x)$ for all x in an interval I, we say that $F(x)$ is an antiderivative for $f(x)$ on I.

Antiderivatives

Example: Consider $f(x) = x^2$.

Let $F(x) = \frac{x^3}{3}$. Then

$$F'(x) = \frac{3x^{3-1}}{3} = x^2 = f(x),$$

so $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x)$.

Note: Notice that

$$G(x) = \frac{x^3}{3} + 2$$

is also an antiderivative of $f(x) = x^2$.

Antiderivatives

Recall: If a function $h(x)$ is constant on an open interval I, then $h'(x) = 0$ for all $x \in I$.

Important Observation: Given any function $f(x)$, if $F(x)$ is an antiderivative of $f(x)$, then so is

$$G(x) = F(x) + C$$

for any $C \in \mathbb{R}$.

Question: Are all antiderivatives of $f(x)$ of the form

$$G(x) = F(x) + C$$

for some $C \in \mathbb{R}$?

Constant Function Theorem

Theorem: [Constant Function Theorem]

Assume that $f'(x) = 0$ for all $x \in I$, then there exists a $\alpha \in \mathbb{R}$ such that $f(x) = \alpha$ for every $x \in I$.

Proof: Let x_1 be any point in I and let

$$f(x_1) = \alpha.$$

Pick any other $x_2 \in I$. Then the MVT guarantees us that there exists a c between x_1 and x_2 with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Then since $f'(c) = 0$, we have

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and hence

$$f(x_2) = f(x_1) = \alpha.$$

Constant Function Theorem

Observation: We know that if $f(x) = e^x$, then

$$f(x) = f'(x)$$

for all $x \in \mathbb{R}$.

Question: Are there other functions $g(x)$ with

$$g(x) = g'(x)?$$

Constant Function Theorem

Example: Show that if $g(x)$ is such that

$$g(x) = g'(x)$$

for all $x \in \mathbb{R}$, there exists a $C \in \mathbb{R}$ so that

$$g(x) = Ce^x.$$

Solution: Let

$$h(x) = \frac{g(x)}{e^x}.$$

Differentiate $h(x)$ using the quotient rule to get

$$\begin{aligned} h'(x) &= \frac{e^x g'(x) - \frac{d}{dx}(e^x) g(x)}{(e^x)^2} \\ &= \frac{e^x g(x) - e^x g(x)}{e^{2x}} \\ &= 0 \end{aligned}$$

since $g'(x) = g(x)$ and $\frac{d}{dx}(e^x) = e^x$. So there exists $C \in \mathbb{R}$ with

$$h(x) = \frac{g(x)}{e^x} = C \Rightarrow g(x) = Ce^x.$$

Antiderivatives

Theorem: [The Antiderivative Theorem]

Assume that $f'(x) = g'(x)$ for all $x \in I$. Then there exists an α such that

$$f(x) = g(x) + \alpha$$

for every $x \in I$.

Proof: Let

$$H(x) = f(x) - g(x).$$

Then

$$H'(x) = f'(x) - g'(x) = 0$$

for each $x \in I$. Therefore, there exists $\alpha \in \mathbb{R}$ so that

$$H(x) = \alpha \Rightarrow f(x) = g(x) + \alpha$$

for all $x \in I$. ◀

Indefinite Integrals

Leibniz Notation for Antiderivatives:

We will denote the *family of antiderivatives* of a function $f(x)$ by

$$\int f(x)\,dx.$$

For example,

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

The symbol

$$\int f(x)\,dx$$

is called the *indefinite integral of* $f(x)$. The function $f(x)$ is called the *integrand*.

Indefinite Integrals

Theorem: [Power Rule for Antiderivatives]

If $\alpha \neq -1$, then

$$\int x^{\alpha}\, dx = \frac{x^{\alpha+1}}{\alpha+1} + C.$$

Note: To check that this theorem is correct we need only differentiate.

Since

$$\frac{d}{dx}\left(\frac{x^{\alpha+1}}{\alpha+1} + C\right) = x^{\alpha},$$

we have found all of the antiderivatives of x^{α}.

Indefinite Integrals

Examples:

1)
$$\int \frac{1}{x}\, dx = \ln(|\, x\,|) + C.$$

2)
$$\int e^x\, dx = e^x + C.$$

3)
$$\int \sin(x)\, dx = -\cos(x) + C.$$

4)
$$\int \cos(x)\, dx = \sin(x) + C.$$

5)
$$\int \sec^2(x)\, dx = \tan(x) + C.$$

Indefinite Integrals

Examples:

6) $$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C.$$

7) $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C.$$

8) $$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C.$$

Remark: It can be shown that there is no *nice* function that represents

$$\int e^{x^2}\,dx.$$