Newton's Method

Created by

Barbara Forrest and Brian Forrest

Newton's Method

Problem Revisited: Given a function $f(x)$ solve the equation

$$f(x) = 0. \quad (*)$$

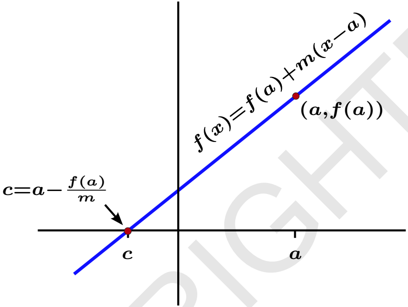
Recall: If $f(x)$ is continuous on $[a, b]$ and if $f(a) \cdot f(b) < 0$, then the *Intermediate Value Theorem* shows that there exists $c \in (a, b)$ with

$$f(c) = 0.$$

Moreover, the *Bisection Method* provides us with an algorithm to approximate c as closely as we choose.

Question:
Do better algorithms exist for finding approximate solutions to $(*)$?

Newton's Method



Motivating Example: Assume that the graph of $f(x)$ is a line through the point $(a, f(a))$ with slope $m \neq 0$. Find the point c where

$$f(c) = 0.$$

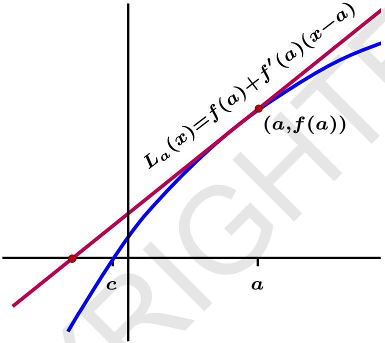
Note that $f(x) = f(a) + m(x - a)$. Therefore,

$$0 = f(c) = f(a) + m(c - a) \Rightarrow -f(a) = m(c - a).$$

Finally,

$$c = a - \frac{f(a)}{m} = a - \frac{f(a)}{f'(a)}.$$

Newton's Method



Question: What can we do if the graph of $f(x)$ is not a line?

Key Idea: Suppose that $x = a$ is close to the solution c where

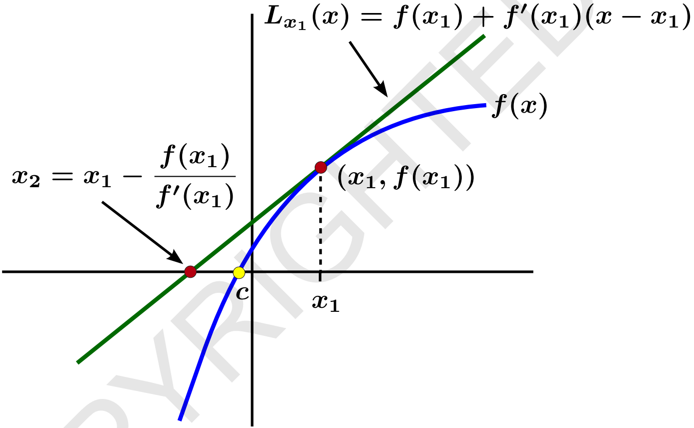
$$f(c) = 0.$$

If $f(x)$ is differentiable at $x = a$ with $f\,'(a) \neq 0$, then
$L_a(x) = f(a) + f\,'(a)(x - a)$ and since

$$f(x) \cong L_a(x)$$

the graphs of $f(x)$ and $L_a(x)$ should cross the x-axis at approximately the same location.

Newton's Method



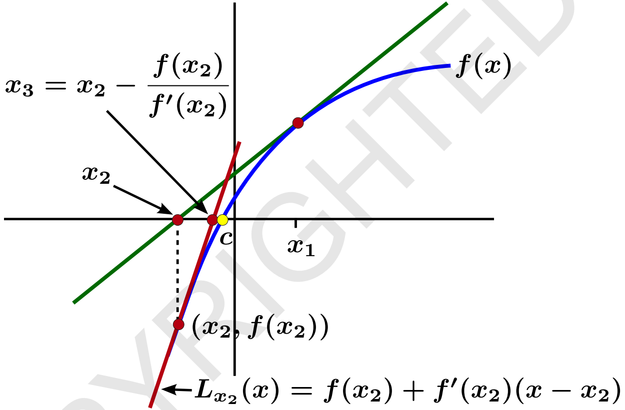
Step 1: Pick an x_1 as close as possible to the point c with $f(c) = 0$.
Step 2: If $f'(x_1) \neq 0$, we can approximate c by x_2, where x_2 is such that

$$L_{x_1}(x_2) = 0.$$

Then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Newton's Method



Step 3: Repeat **Step 2** to get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

This gives a recursive sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Newton's Method

Convergence of Newton's Method

1) For most **nice** functions and reasonable choices of x_1, the sequence $\{x_n\}$ converges very rapidly to a number c with $f(c) = 0$.
2) Typically the number of decimal places of accuracy in Newton's Method doubles with each iteration (quadratic convergence). The Bisection Method takes roughly 4 iterations to improve accuracy by one decimal place.
3) To achieve n-decimal places of accuracy, terminate the procedure when two consecutive iterations agree to n-decimal places.
4) Unlike the Bisection Method, Newton's Method can fail to coverge.

Heron's Algorithm Revisited

Problem: Use Newton's method to approximate $\sqrt{2}$ to nine decimal places.

Solution: We must solve

$$f(x) = x^2 - 2 = 0.$$

Step 1: Since we are looking for positive root we let

$$x_1 = 1.$$

Step 2: Determine the recursive sequence:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{(x_n^2 - 2)}{2x_n} \\ &= \frac{1}{2}(x_n + \frac{2}{x_n}) \end{aligned}$$

Note: This is the same recursive sequence that we had in Heron's Algorithm for finding $\sqrt{2}$.

Heron's Algorithm Revisited

Using $x_1 = 1$, we have

$$\begin{aligned} x_2 &= x_{1+1} \\ &= \frac{1}{2}(1 + \frac{2}{1}) \\ &= \frac{1}{2}(3) \\ &= 1.5 \end{aligned}$$

Next we have

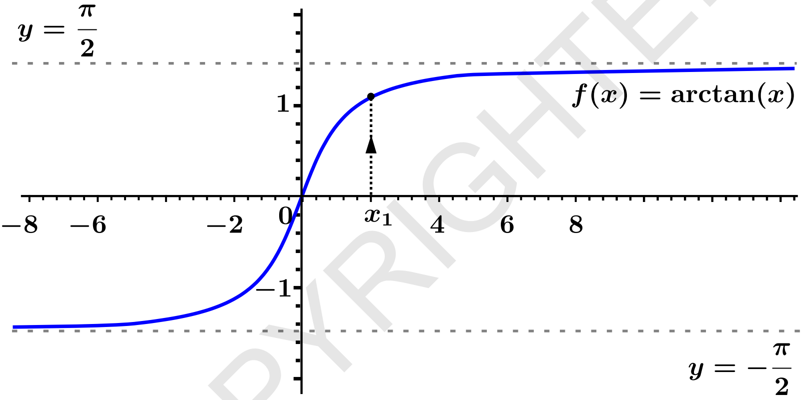
$$\begin{aligned} x_3 &= x_{2+1} \\ &= \frac{1}{2}(\frac{3}{2} + \frac{2}{\frac{3}{2}}) \\ &= \frac{17}{12} \\ &= 1.416666667 \end{aligned}$$

Heron's Algorithm Revisited

Continuing the calculations we get:

$$\begin{aligned}
x_4 &= x_{3+1} \\
&= \frac{1}{2}\left(\frac{17}{12} + \frac{2}{\frac{17}{12}}\right) \\
&= \frac{577}{408} \\
&= 1.414215686 \\
x_5 &= x_{4+1} \\
&= \frac{1}{2}\left(\frac{577}{408} + \frac{2}{\frac{577}{408}}\right) \\
&= \frac{665857}{470832} \\
&= 1.414213562 \\
x_6 &= x_{5+1} \\
&= \frac{1}{2}\left(\frac{665857}{470832} + \frac{2}{\frac{665857}{470832}}\right) \\
&= 1.414213562 \cong \sqrt{2}
\end{aligned}$$

Failure of Newton's Method

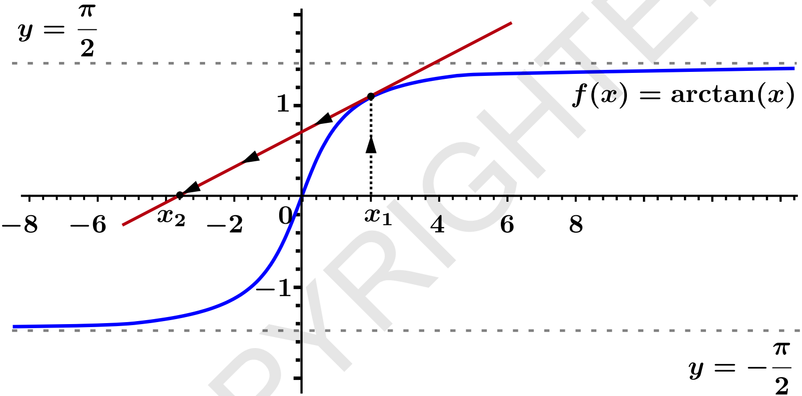


Example: Let

$$f(x) = \arctan(x)$$

and $|x_1| > 1.4$.

Failure of Newton's Method



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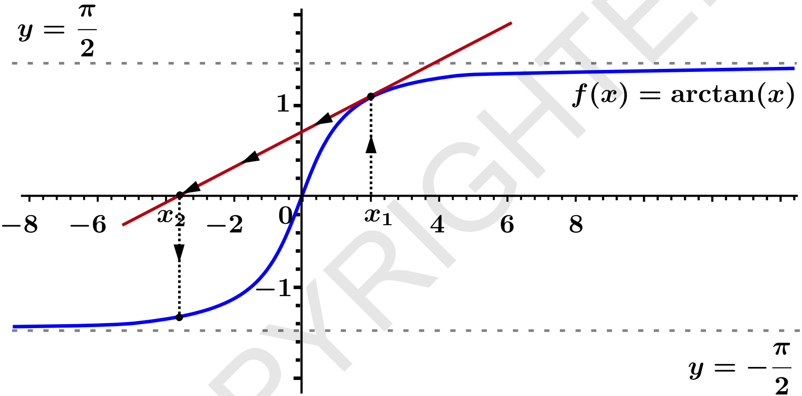

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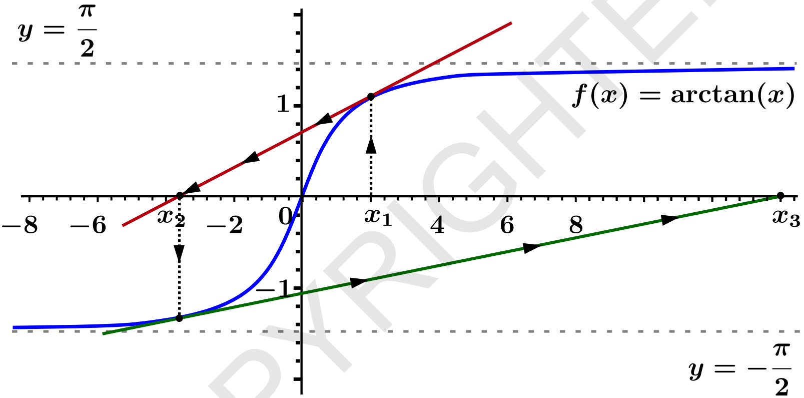

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