More Trigonometric Derivatives

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Derivative of $\sin(x)$

Theorem: [The Derivative of $\sin(x)$]

Assume that $f(x) = \sin(x)$. Then

$$f'(x) = \cos(x)$$

for all $x \in \mathbb{R}$.

Question: Can we use the properties of derivatives to find the derivative of other trigonometric functions?

Derivative of $\cos(x)$

Example: Find $\frac{d}{dx}(\cos(x))$.

Solution: We know that

$$\cos(x) = \sin(x + \frac{\pi}{2}).$$

Let $y = \sin(u)$ and $u = x + \frac{\pi}{2}$. Substituting for u gives us that

$$y = y(x) = \sin(x + \frac{\pi}{2}) = \cos(x).$$

Therefore,

$$\begin{aligned}
\frac{d}{dx}(\cos(x)) &= \frac{dy}{dx} \\
&= \frac{dy}{du}\frac{du}{dx} \\
&= \cos(u) \cdot (1) \\
&= \cos(x + \frac{\pi}{2}) \\
&= \cos(x)\cos(\frac{\pi}{2}) - \sin(x)\sin(\frac{\pi}{2}) \\
&= -\sin(x).
\end{aligned}$$

Derivative of $\tan(x)$

Example: Find $\frac{d}{dx}(\tan(x))$.

Solution: Since

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

we get

$$\begin{aligned}
\frac{d}{dx}(\tan(x)) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \frac{\left(\frac{d}{dx}\sin(x)\right)\cos(x) - (\sin(x))\left(\frac{d}{dx}\cos(x)\right)}{\cos^2(x)} \\
&= \frac{\cos(x)\cos(x) - (\sin(x))(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x).
\end{aligned}$$

Derivative of $\cot(x)$

Note: We have just seen that

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

whenever $\cos(x) \neq 0$. A similar calculation shows that

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

whenever $\sin(x) \neq 0$.

Derivative of $\sec(x)$ and $\csc(x)$

Example: Find $\frac{d}{dx}(\sec(x))$.

Solution: Since $\sec(x) = \frac{1}{\cos(x)}$, we get

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) &= -\frac{\frac{d}{dx}(\cos(x))}{\cos^2(x)} \\ &= \frac{-(-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} \\ &= \tan(x)\sec(x). \end{aligned}$$

A similar calculation shows that

$$\frac{d}{dx}(\csc(x)) = -\cot(x)\csc(x).$$