Local Extrema

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Global Extrema

Definition: [Global Extrema]

Suppose that $f(x)$ is defined on some set S.

1) We say that $f(x)$ has a *global maxima* on S at $x = c$ if

$$f(x) \leq f(c)$$

for every $x \in S$.

2) We say that $f(x)$ has a *global minimum* on S at $x = c$ if

$$f(c) \leq f(x)$$

for every $x \in S$.

Global Extrema

Recall: The *Extreme Value Theorem* tells us that if $f(x)$ is continuous on $[a, b]$, then there exists $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$, and each of c and d is either

1) an endpoint, or
2) is in the open interval (a, b).

Local Extrema

Definition: [Local Extrema]

1) We say that $f(x)$ has a *local maximum* at $x = c$ if there exists an open interval (a, b) containing c such that

$$f(x) \leq f(c)$$

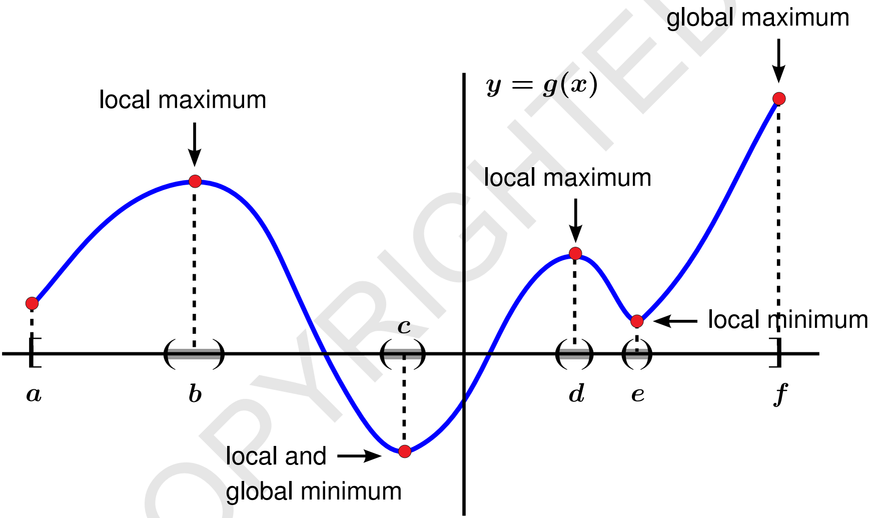
for every $x \in (a, b)$.

2) We say that $f(x)$ has a *local minimum* at $x = c$ if there exists an open interval (a, b) containing c such that

$$f(c) \leq f(x)$$

for every $x \in (a, b)$.

Local and Global Extrema



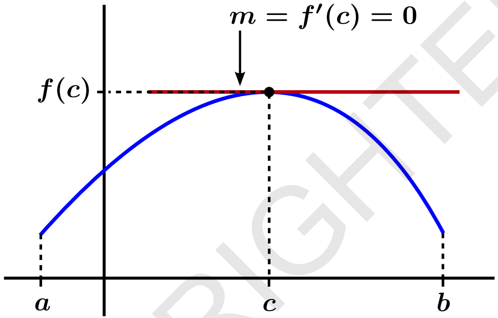
$g(x)$ is defined on $[a, f]$.

Local Extrema

Problem:

If f has either a local maximum or local minimum at $x = c$ and f is differentiable at $x = c$, what can we say about $f'(c)$?

Local Extrema



Solution: Assume that $f(x)$ has a local maximum at $x = c$ and that $c \in (a, b)$ with $f(x) \leq f(c)$ for all $x \in (a, b)$. Assume that $f'(c)$ exists.

1. If $a < c + h < c$, then

   $$\frac{f(c+h)-f(c)}{h} \geq 0 \Rightarrow f'(c) = \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0.$$

2. If $c < c + h < b$, then

   $$\frac{f(c+h)-f(c)}{h} \leq 0 \Rightarrow f'(c) = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0.$$

Hence $f'(c) = 0$.

Local Extrema

Theorem: [Local Extrema Theorem]

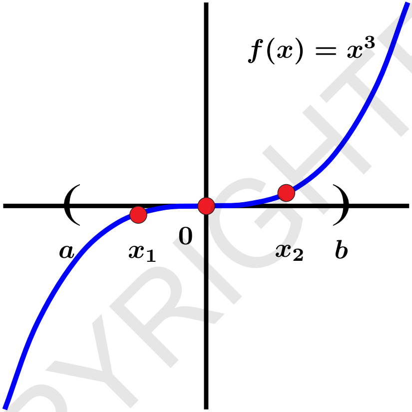
Assume that $f(x)$ has either a local maximum or a local minimum at $x = c$. If $f(x)$ is differentiable at $x = c$, then

$$f'(c) = 0.$$

Question:

If $f'(c) = 0$ must $x = c$ be either a local maximum or local minimum?

Local Extrema

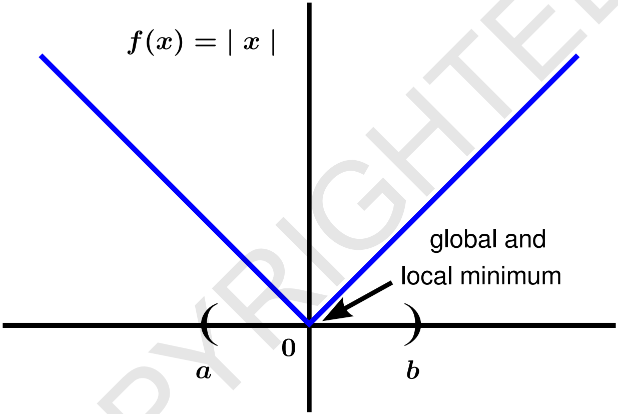


Example: Let $f(x) = x^3$. Then $f'(x) = 3x^2$ so that

$$f'(0) = 0.$$

But $f(x)$ has neither a local maximum nor a local minimum at $x = c$ since $f(x)$ is always increasing.

Local Extrema



Question: If $f(x)$ has a local maximum or minimum at $x = c$, must $f'(c) = 0$?

Example: Let $f(x) = |x|$. Then $f(x)$ has a local (and global) minimum at $x = 0$, but $f'(0)$ does not exist.

Critical Point

Definition: [Critical Point]

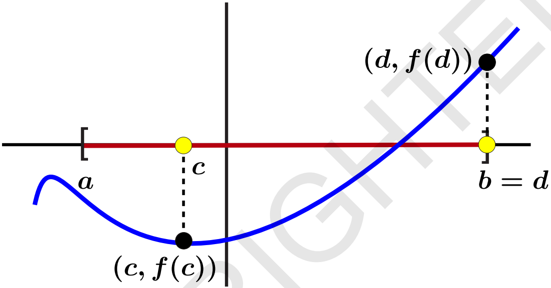
A point c in the domain of a function $f(x)$ is called a *critical point* for $f(x)$ if either

$$f'(c) = 0$$

or

$$f'(c) \text{ does not exist.}$$

Global Extrema and the EVT



Theorem: [Extreme Value Theorem (EVT)]

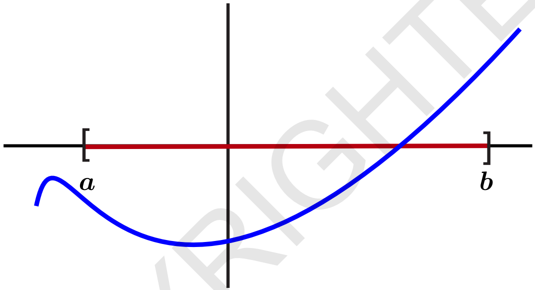
Assume that $f(x)$ is continuous on $[a, b]$. Then there exists $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for every $x \in [a, b]$.

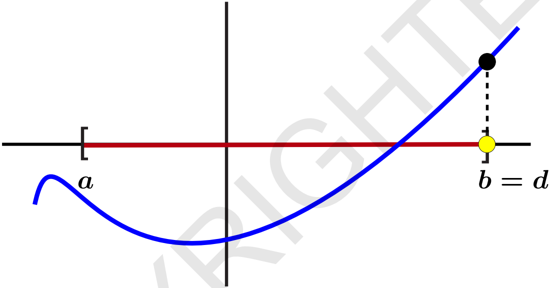
Question: How can we find c and d?

Global Extrema and the EVT



Two Cases:

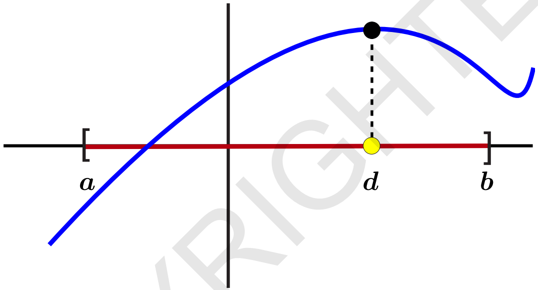
Global Extrema and the EVT



Two Cases:

1) d is an endpoint.

Global Extrema and the EVT



Two Cases:

1) d is an endpoint.
2) $d \in (a, b) \Rightarrow d$ is a local max$\Rightarrow$ d is a critical point.

   Similarly for c.

Finding Local Extrema

Question:

How do we know if a critical point $x = c$ is actually a local maximum or a local minimum?

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How do we know if a critical point $x = c$ is actually a local maximum or a local minimum?