Linear Approximation: Basics

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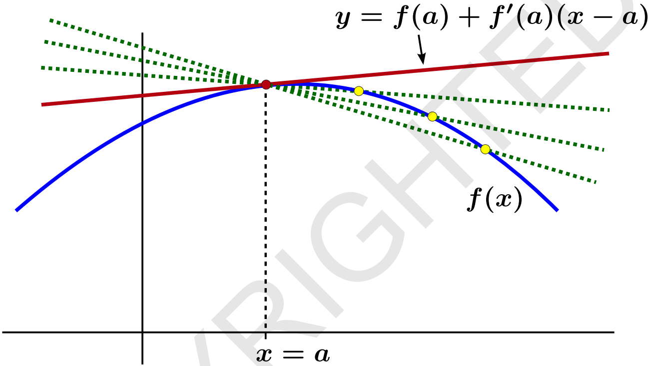
Tangent Line



Question: Does there exist a geometric interpretation of the derivative?

Definition: [Tangent Line]

The *tangent line to the graph of* $f(x)$ *at* $x = a$ is the line passing through $(a, f(a))$ with slope equal to $f'(a)$. That is

$$y = f(a) + f'(a)(x-a).$$

Linear Approximation

Fundamental Observation:

Suppose that $f(x)$ is differentiable at $x = a$ with derivative $f'(a)$. Then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Hence *for values of x that are close to a* we have

$$\frac{f(x) - f(a)}{x - a} \cong f'(a). \qquad (*)$$

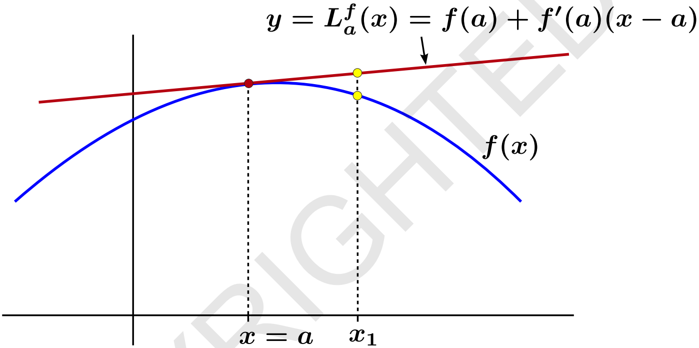
Rearranging $(*)$, we get

$$f(x) - f(a) \cong f'(a)(x - a)$$

and finally that

$$f(x) \cong f(a) + f'(a)(x - a). \quad (**)$$

Linear Approximation



Definition: [Linear Approximation]

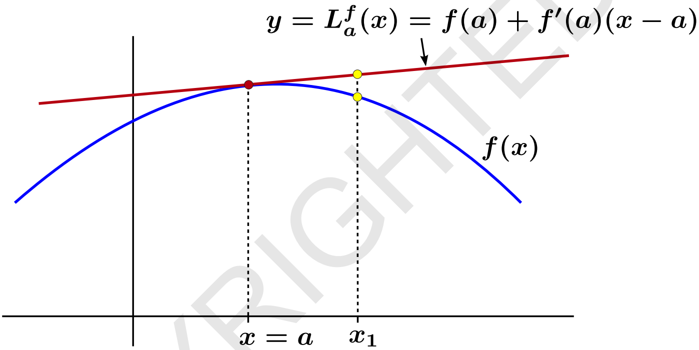
Let $y = f(x)$ be differentiable at $x = a$. The linear approximation to $f(x)$ at $x = a$ is the function

$$L_a^f(x) = f(a) + f'(a)(x-a).$$

$L_a^f(x)$ is also called the ***linearization*** of $f(x)$ or the ***tangent line approximation*** to $f(x)$ at $x = a$.

Note: If $f(x)$ is clear from the context, then we will simply write $L_a(x)$.

Linear Approximation



Summary: If

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

then if $x \cong a$,

$$L_a^f(x) \cong f(x).$$

Observation: The graph of $L_a^f(x)$ is the tangent line to the graph of $f(x)$ through $(a, f(a))$.

Fundamental Properties of $L_a^f(x)$

Three Fundamental Properties of $L_a^f(x)$:

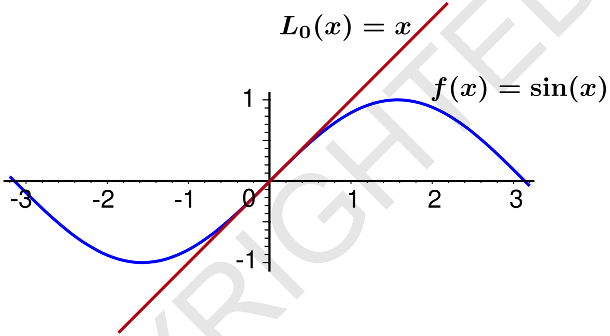
Assume that $f(x)$ is differentiable at $x = a$ with

$$L_a^f(x) = f(a) + f'(a)(x - a).$$

Then:

1) $L_a^f(a) = f(a)$.
2) $L_a^f(x)$ is differentiable at $x = a$ and $L_a^f{}'(a) = f'(a)$.
3) $L_a^f(x)$ is the **only** first degree polynomial with Property (1) and Property (2).

Fundamental Trig Limit



Recall: The Fundamental Trig Limit

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Observe: If $f(x) = \sin(x)$, then $f(0) = 0$ and $f'(0) = \cos(0) = 1$ so

$$\sin(x) \cong L_0(x) = x$$

when $x \cong 0$.

Fundamental Trig Limit

Example: Use linear approximation to estimate $\sin(.01)$.

Solution: We have

$$\sin(.01) \cong L_0(.01) = x\,|_{.01} = .01$$

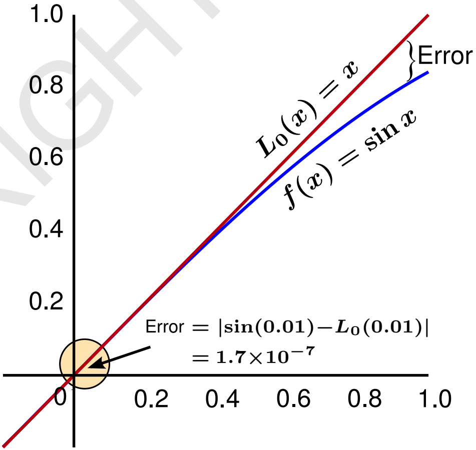
Note: In fact

$$\sin(.01) = .00999983$$

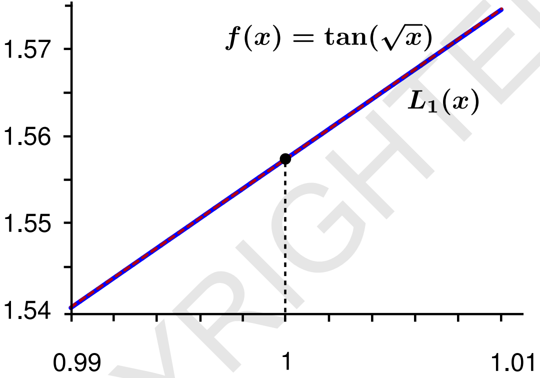
to eight decimal places.

So the error is

$$\begin{aligned} \text{Error} &= |\sin(.01) - L_0(.01)| \\ &\cong .00000017 \\ &= 1.7 \times 10^{-7}. \end{aligned}$$



Example



Example: Let $f(x) = \tan(\sqrt{x})$ and $a = 1$.

Key Observation:

Over very small intervals differentiable functions appear like lines.