Linear Approximation: Applications

Created by

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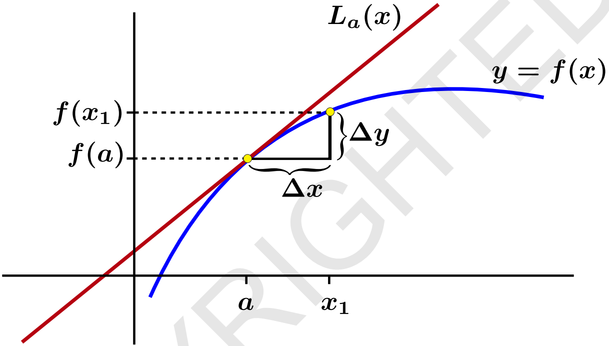
Estimating Change



Problem: Assume that we know the value of a function $f(x)$ at a point a. How do we estimate the change we could expect in the value of $f(x)$ if we move to a point x_1 near a?

That is, we want to estimate the value of

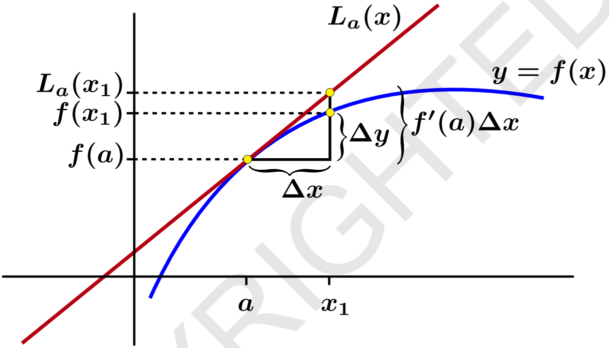
$$\Delta y = f(x_1) - f(a)$$

if we change our variable by

$$\triangle x = x_1 - a$$

units.

Estimating Change



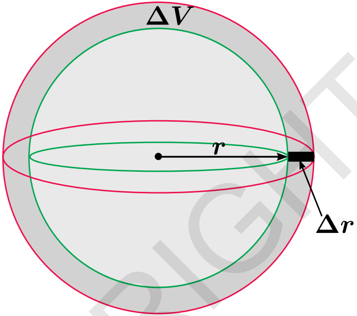
Solution: If we were to use linear approximation, we get that

$$\begin{aligned} \Delta y &= f(x_1) - f(a) \\ &\cong L_a(x_1) - f(a) \\ &= (f(a) + f'(a)(x_1 - a)) - f(a) \\ &= f'(a)(x_1 - a) \\ &= f'(a)\triangle x. \end{aligned}$$

That is,

$$\Delta y \cong f'(a)\triangle x.$$

Estimating Change



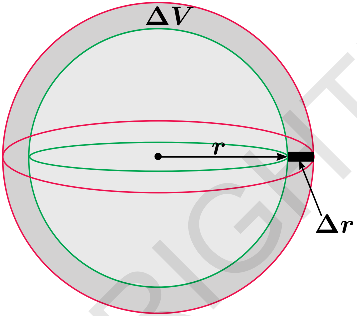
Example: A metal sphere of radius 10 cm expands when heated so that its radius increases by 0.01 cm. Estimate the change in the volume of the sphere.

Solution: We know that the volume (V) of the sphere with radius r is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

and that $V'(r) = 4\pi r^2$.

Estimating Change



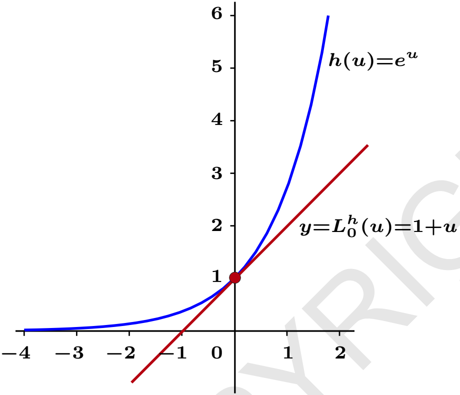
Solution (continued): Our focal point is at $r = 10$, so

$$V'(10) = 400\pi.$$

We also know that $\triangle r = .01$, so

$$\begin{aligned} \Delta V &= V(10.01) - V(10) \\ &\cong V'(10)\triangle r \\ &= 400\pi(.01) \\ &= 4\pi \text{ cm}^3. \end{aligned}$$

Qualitative Analysis of Functions



Problem: How does the function

$$f(x) = e^{-x^2}$$

behave near $x = 0$?

Solution:

Step 1: Start with the simpler function

$$h(u) = e^u.$$

Since

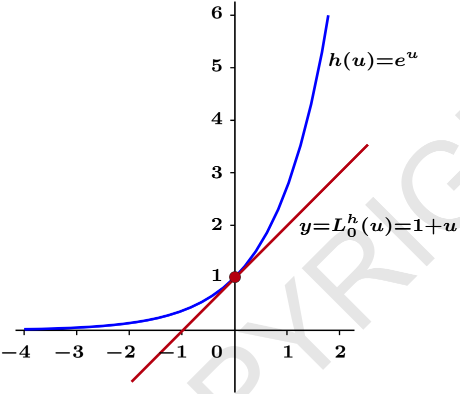
$$h(0) = h'(0) = e^0 = 1$$

we get that

$$e^u \cong L_0^h(u) = 1 + u$$

so long as u is near 0.

Qualitative Analysis of Functions



Problem: How does the function

$$f(x) = e^{-x^2}$$

behave near $x = 0$?

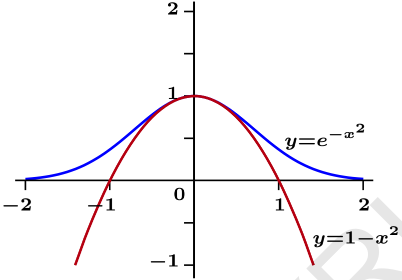
Solution (continued):

Step 2: We know that

$$e^u \cong L_0^h(u) = 1 + u$$

so long as u is near 0.

Qualitative Analysis of Functions



Problem: How does the function

$$f(x) = e^{-x^2}$$

behave near $x = 0$?

Solution (continued):

Step 2: We know that

$$e^u \cong L_0^h(u) = 1 + u$$

so long as u is near 0.

If x is close to 0, then so is $u = -x^2$. Letting $u = -x^2$, we get

$$y = e^{-x^2} \cong 1 + (-x^2) = 1 - x^2$$

if $x \cong 0$.