Linear Approximation: The Error

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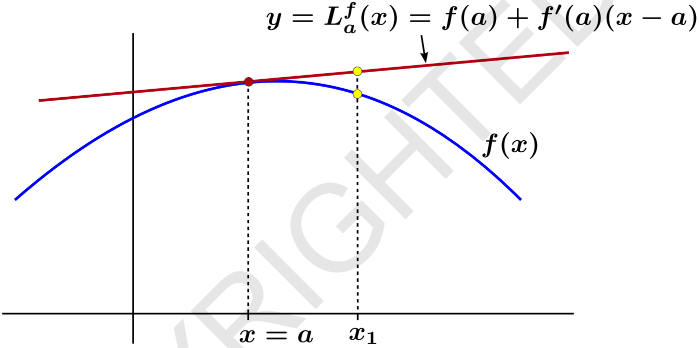
Linear Approximation



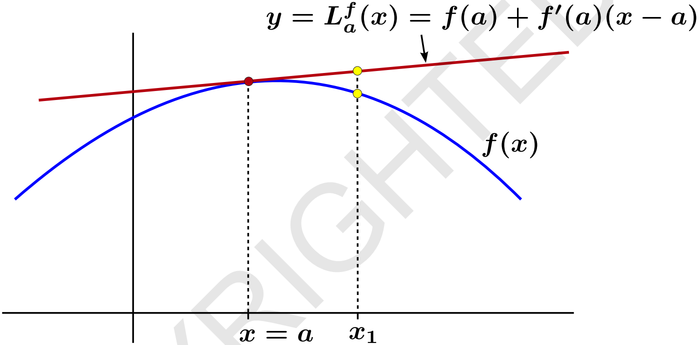
Definition: [Linear Approximation]

Let $y = f(x)$ be differentiable at $x = a$. The linear approximation to $f(x)$ at $x = a$ is the function

$$L_a^f(x) = f(a) + f'(a)(x - a).$$

$L_a^f(x)$ is also called the ***linearization*** of $f(x)$ or the ***tangent line approximation*** to $f(x)$ at $x = a$.

Linear Approximation



Key Observation: If

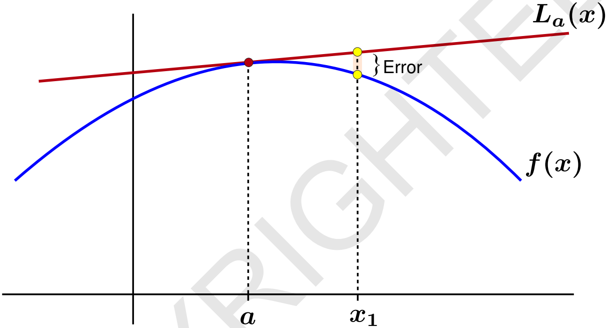
$$L_a^f(x) = f(a) + f\,'(a)(x - a)$$

then if $x \cong a$,

$$L_a^f(x) \cong f(x).$$

Note: If $f(x)$ is clear from the context, then we will simply write $L_a(x)$.

The Error in Linear Approximation



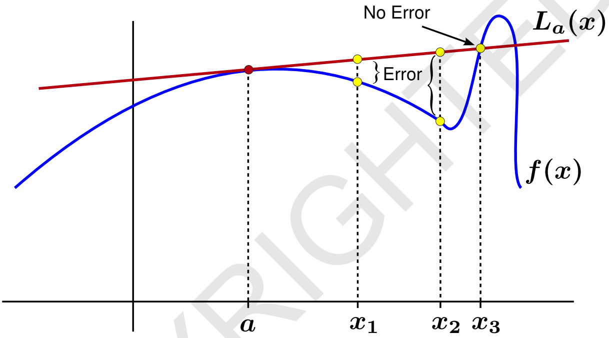
Definition: [The Error in Linear Approximation]

Let $y = f(x)$ be differentiable at $x = a$. The error in using linear approximation to estimate $f(x)$ is given by

$$\text{Error} = | \ f(x) - L_a(x) \ | \ .$$

Question: What are the key factors affecting the magnitude of the error?

The Error in Linear Approximation



First Observation: Since the approximation

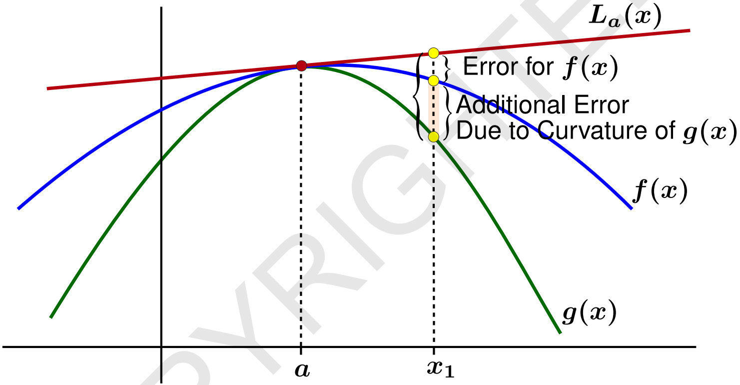
$$f(x) \cong f(a) + f'(a)(x-a) = L_a(x)$$

was obtained from the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

the further we are from a, the larger the potential error.

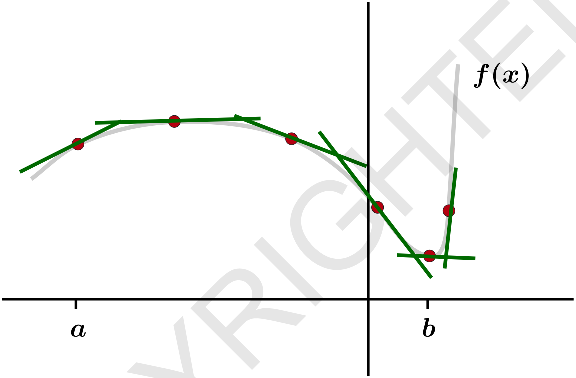
The Error in Linear Approximation



Second Observation: Since we are using a line to approximate the graph of the function, the more *curved* the graph is near $x = a$, the greater the potential error.

Problem:
How can we quantify the phrase “the more curved the graph is?”

Curvature and the Second Derivative



Problem: To quantify the phrase “the more curved the graph is.”

Solution: Curvature arises from a change in the slope of the tangent lines. The more quickly these slopes change, the more curved the graph.

Conclusion: The larger the magnitude of $f''(x)$ near $x = a$, the greater the curvature of the graph and the larger the potential error in using linear approximation.

The Error in Linear Approximation

Theorem: [The Error in Linear Approximation]

Assume that $f(x)$ is such that $\mid f''(x) \mid \leq M$ for each x in an interval I containing a point a. Then

$$\mid f(x) - L_a(x) \mid \leq \frac{M}{2}(x-a)^2$$

for each $x \in I$.

The Error in Linear Approximation

Example: We saw that if $f(x) = \sin(x)$, then

$$L_0(x) = x$$

so

$$\sin(.01) \cong L_0(.01) = .01.$$

Since $|\sin(x)| \leq |x|$, we have

$$|f''(x)| = |-\sin(x)| \leq |x| \leq .01$$

for each $x \in [-.01, .01]$.

The *Error in Linear Approximation Theorem* tells us that

$$\begin{aligned} |\sin(.01) - L_0(.01)| &\leq \frac{M}{2}(.01-0)^2 \\ &= \frac{.01}{2}(.01)^2 \\ &= 5 \times 10^{-7}. \end{aligned}$$