Implicit Differentiation

Created by

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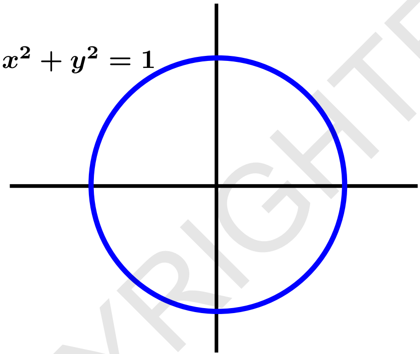
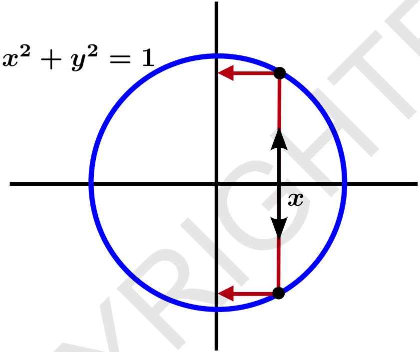
Relations and Implicit Functions



Example: Consider the relation

$$\{(x, y) |\ x^2 + y^2 = 1\}.$$

Relations and Implicit Functions



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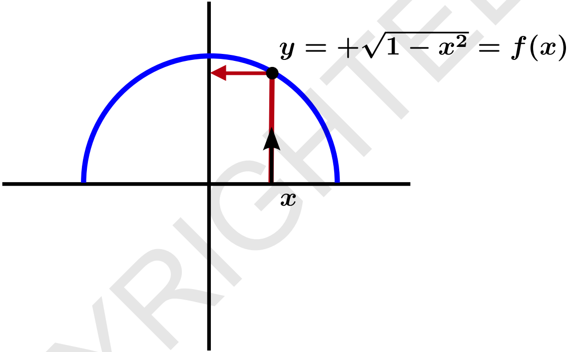
Note: This relation is not a function.

Relations and Implicit Functions



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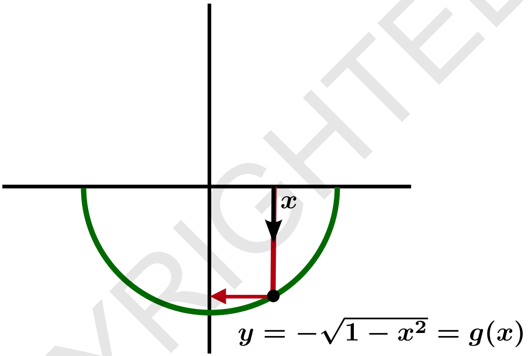
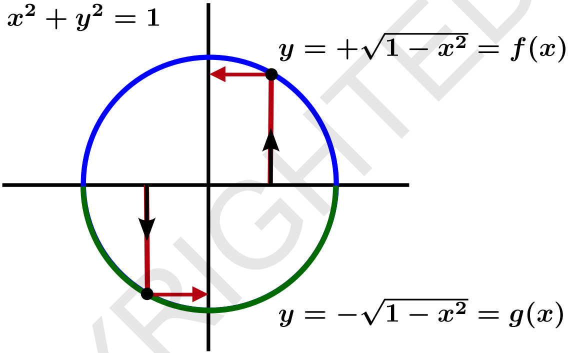


Example: Consider the relation

$$\{(x, y) |\ x^2 + y^2 = 1\}.$$

Note: This relation is not a function. However, by restricting the range of the relation we can generate many functions. We say these functions are *implicitly* defined by the relation.

Relations and Implicit Functions



Example (continued): The relation

$$\{(x, y) |\, x^2 + y^2 = 1\}$$

implies two functions

$$y = f(x) = +\sqrt{1 - x^2} \quad \text{and} \quad y = g(x) = -\sqrt{1 - x^2}$$

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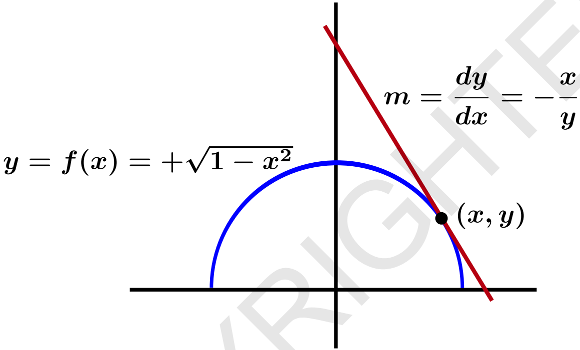

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$$y = f(x) = +\sqrt{1-x^2} \quad \text{and} \quad y = g(x) = -\sqrt{1-x^2}$$

with derivatives

$$f\,'(x) = -\frac{x}{\sqrt{1-x^2}} = -\frac{x}{y}$$

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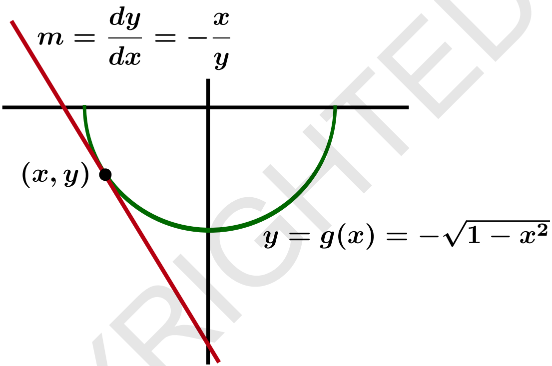

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implies two functions

$$y = f(x) = +\sqrt{1 - x^2} \quad \text{and} \quad y = g(x) = -\sqrt{1 - x^2}$$

with derivatives

$$f'(x) = -\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y} \quad \text{and} \quad g'(x) = \frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y}.$$

Implicit Differentiation

Example (continued): If

$$y = h(x)$$

is any differentiable function which satisfies

$$x^2 + y^2 = 1$$

then

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1).$$

Hence

$$2x + 2y \cdot \frac{dy}{dx} = 0.$$

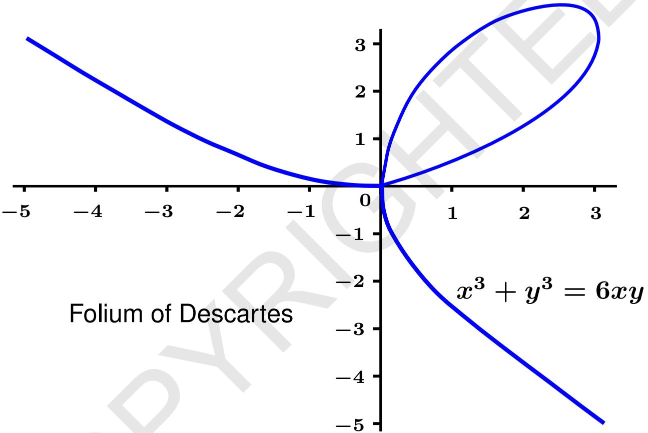
Finally

$$2y \cdot \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

provided that $y \neq 0$.

Note: The process of finding the derivative without knowing the explicit formula for the function is called *implicit differentiation.*

Folium of Descartes

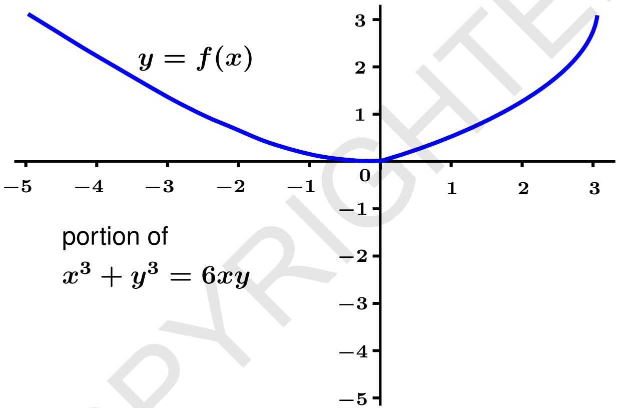


Example: The graph of the relation

$$\{(x, y) | \, x^3 + y^3 = 6xy\}$$

is called the *Folium of Descartes*.

Folium of Descartes



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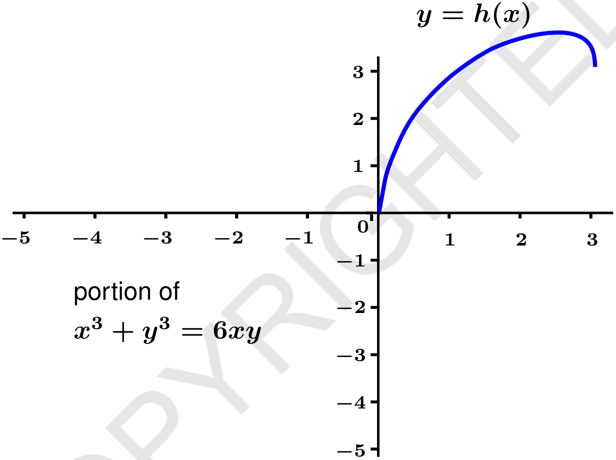

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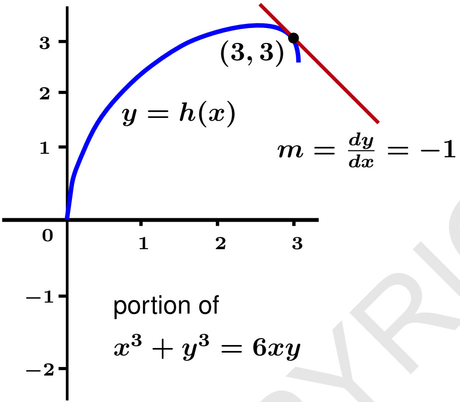

Example: Find $h'(3)$ if $y = h(x)$ is a differentiable function with $h(3) = 3$ satisfying

$$x^3 + y^3 = 6xy.$$

Then

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy).$$

Hence

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

and

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}.$$

Letting $x = 3$ and $y = 3$ gives

$$h'(3) = \left.\frac{dy}{dx}\right|_{(3,3)} = \frac{6(3) - 3(3^2)}{3(3^2) - 6(3)} = -1.$$

Caution!

Example: Suppose that

$$x^4 + y^4 = -1 - x^2y^2. \qquad (*)$$

Find $\dfrac{dy}{dx}$.

Solution: We get

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(-1 - x^2y^2)$$

so

$$\frac{dy}{dx} = \frac{-2xy^2 - 4x^3}{4y^3 + 2x^2y}. \qquad (**)$$

Important Observation:
The equation $(*)$ has no solutions, so $(**)$ is meaningless!