Inverse Function Theorem

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Invertibility and Differentiability

Problem: If $f(x)$ is invertible with inverse $g(y)$ and if $f(x)$ is differentiable at $x = a$, what can we say about the differentiablity of $g(y)$ at $b = f(a)$?

Answer: We will see using the idea of linear approximations that

$$g'(b) = \frac{1}{f'(a)}$$

provided that $f'(a) \neq 0$.

Invertibility and Differentiability

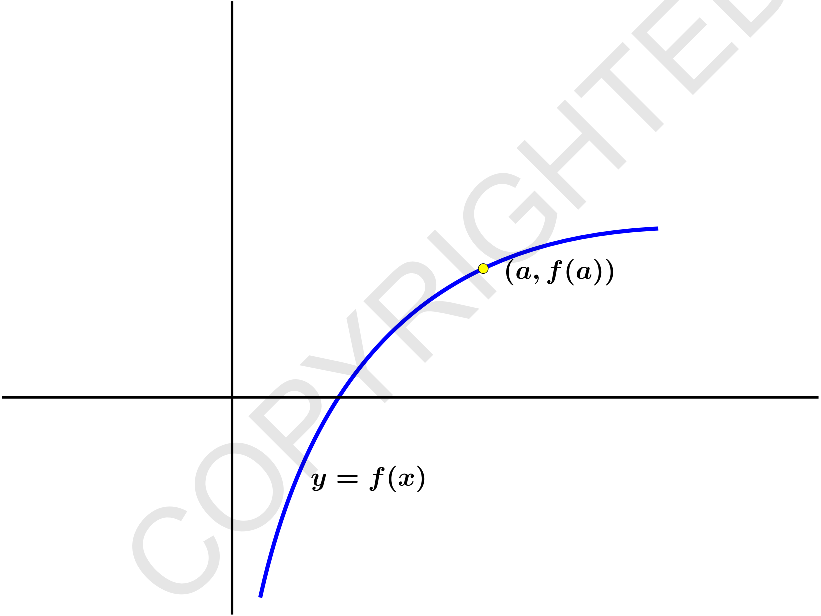
Observe: Given that $f(x)$ is differentiable at $x = a$ we have

$$y = L_a^f(x) = f(a) + f'(a)(x - a)$$

If $f'(a) \neq 0$, then $L_a^f(x)$ is invertible with

$$\begin{aligned}(L_a^f)^{-1}(x) &= a + \frac{1}{f'(a)}(x - f(a)) \\ &= g(f(a)) + \frac{1}{f'(a)}(x - f(a))\end{aligned}$$

Inverse Function Theorem



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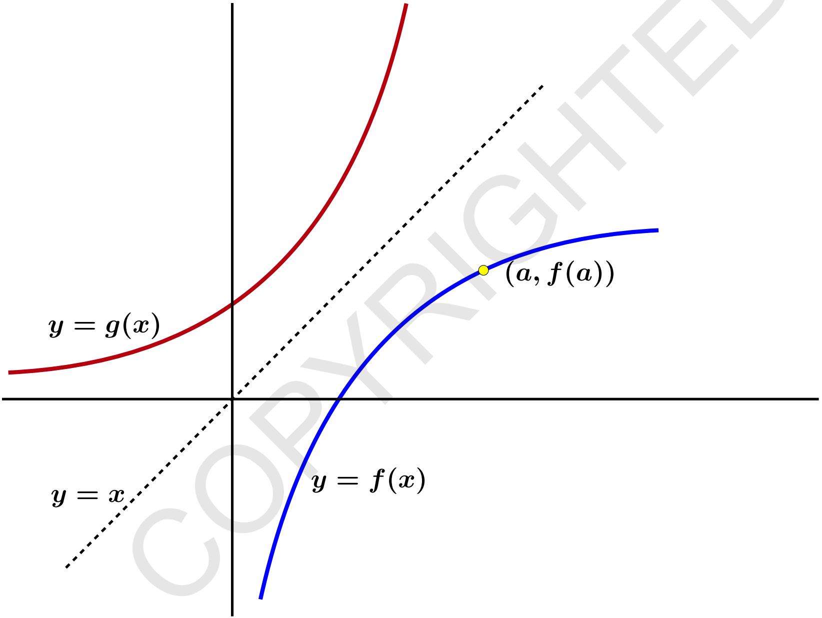

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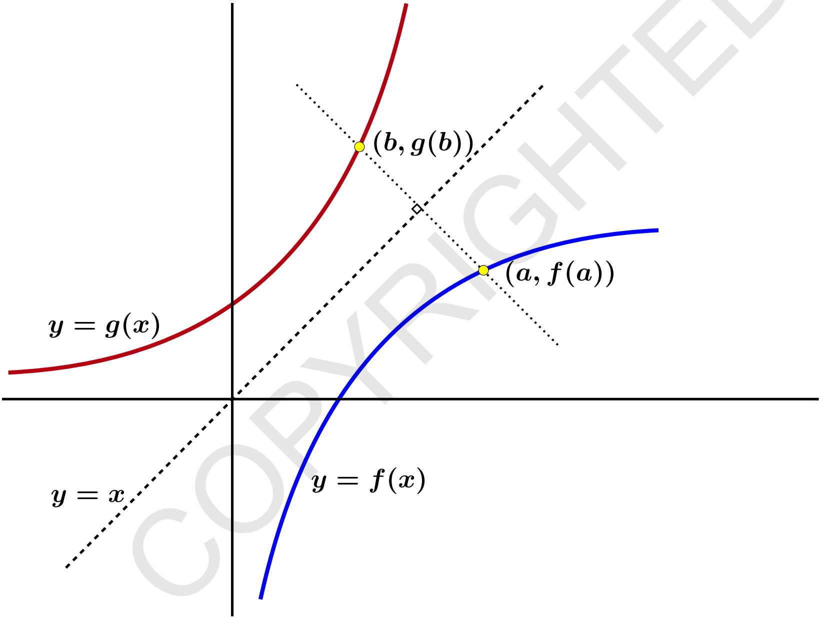

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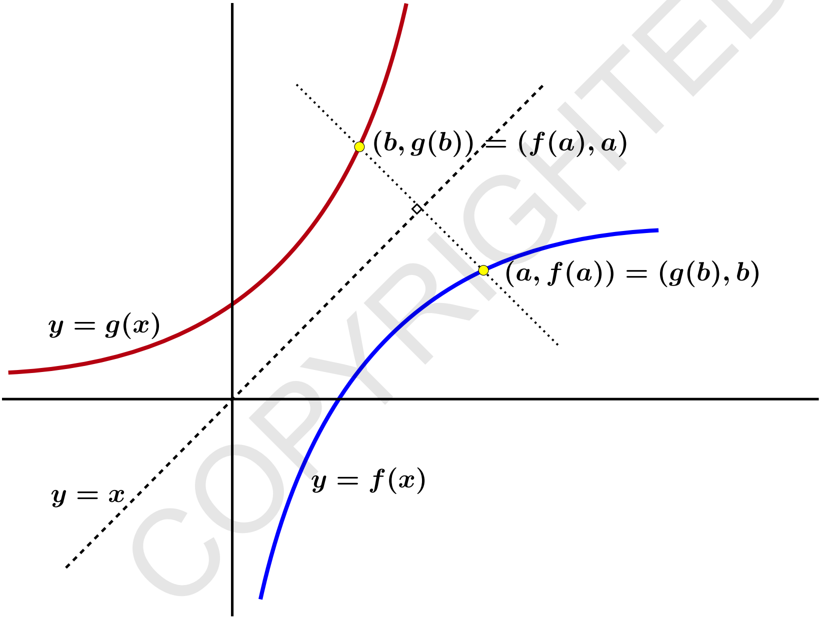

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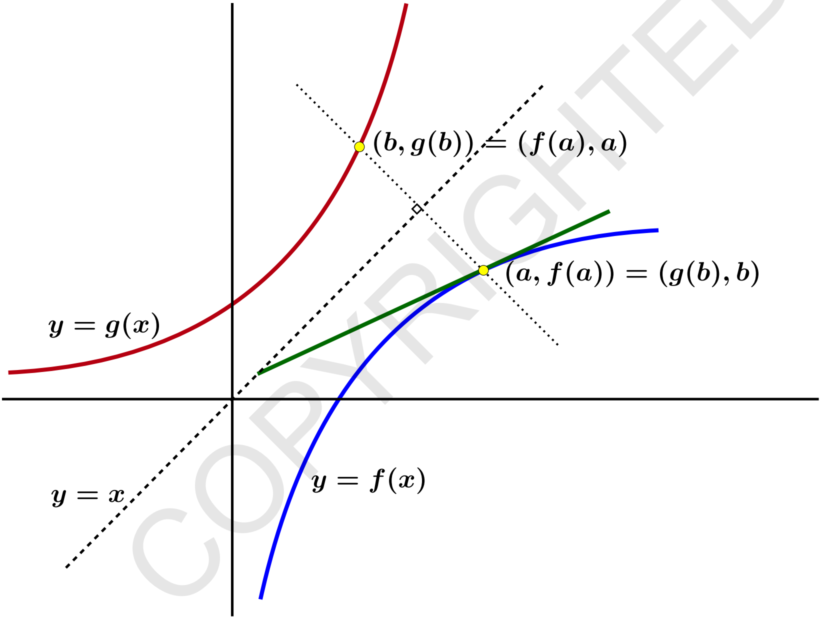

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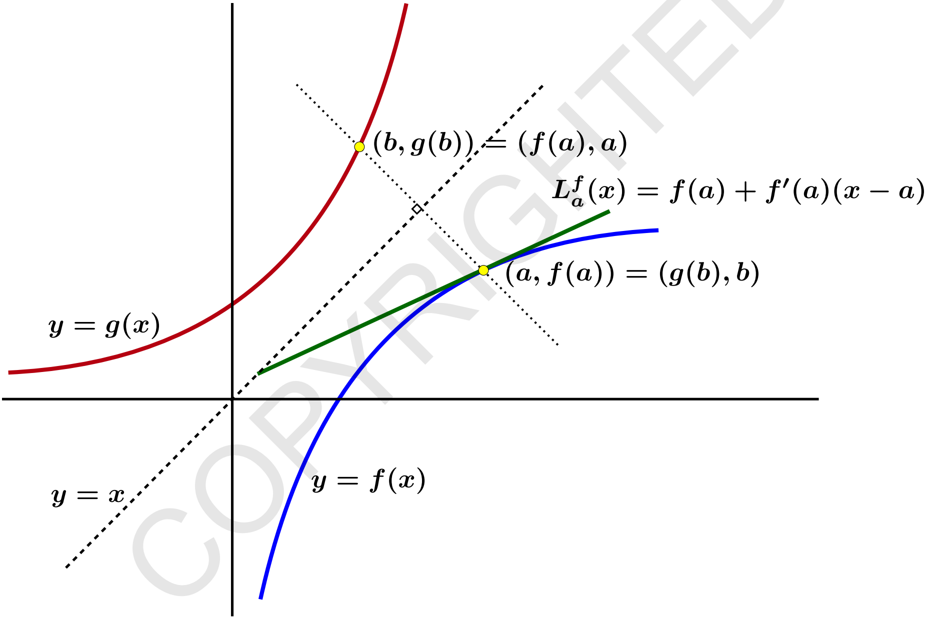

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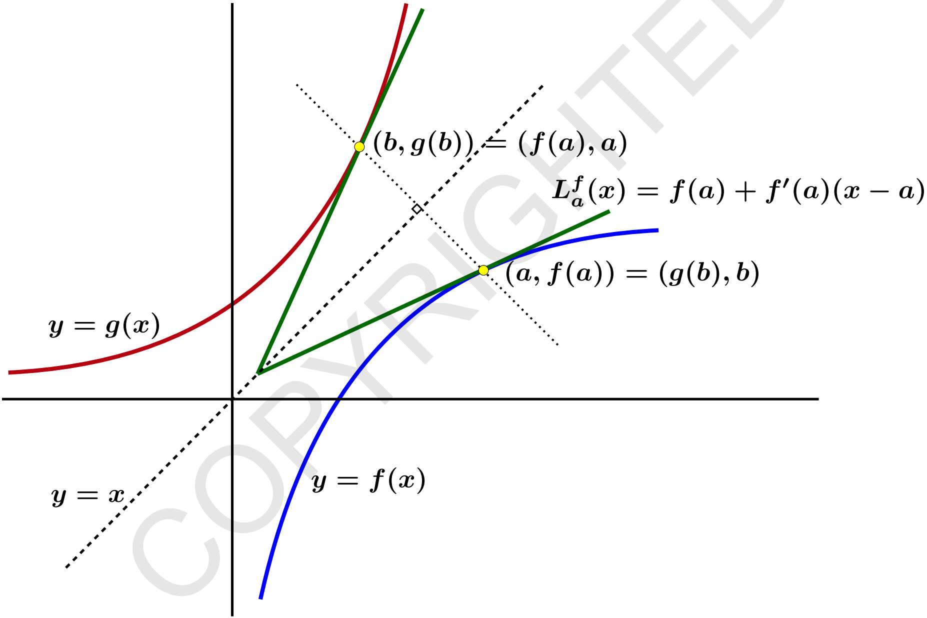

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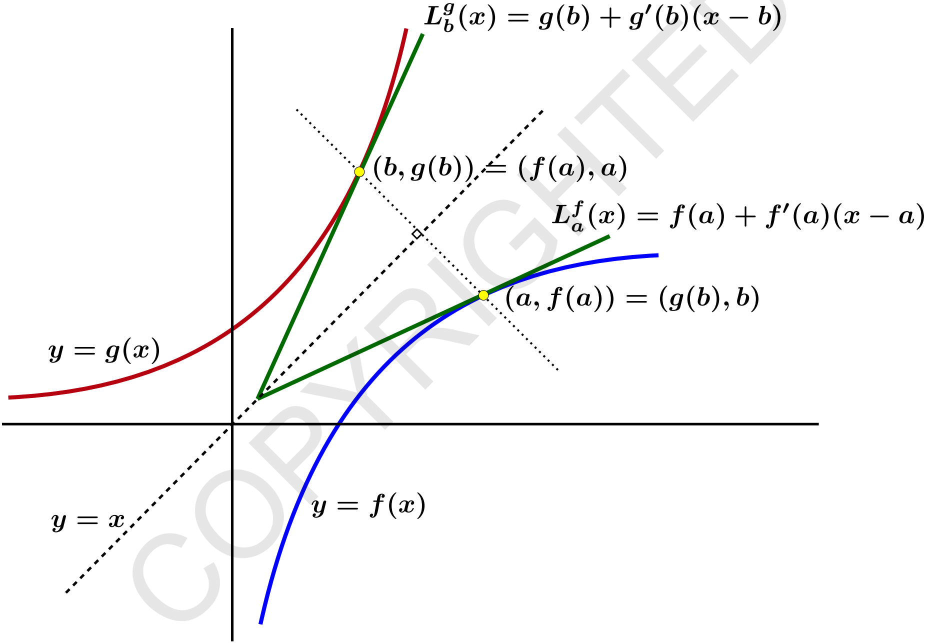

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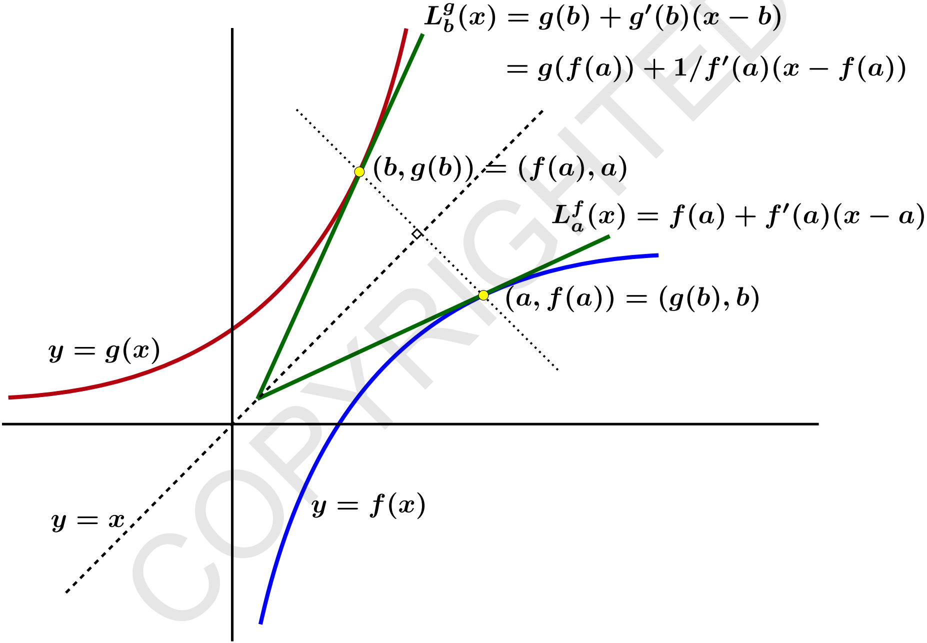

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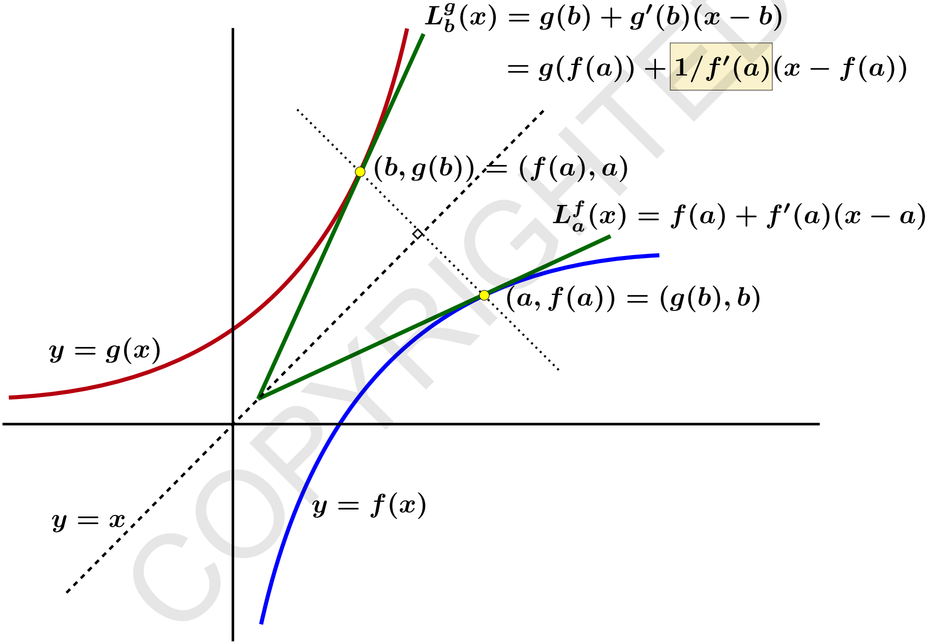

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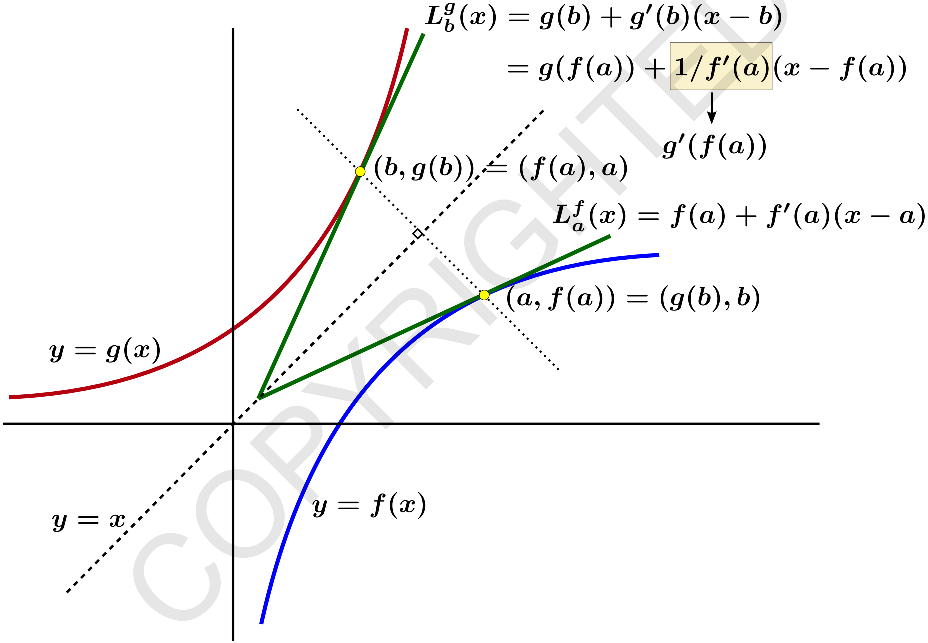


Inverse Function Theorem

Theorem: [Inverse Function Theorem (IFT)]

Assume that $f(x)$ is continuous and invertible on $[c, d]$ with inverse $g(y)$, and $f(x)$ is differentiable at $a \in (c, d)$. If $f'(a) \neq 0$, then $g(y)$ is differentiable at $b = f(a)$, and

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

Moreover, $L_a^f(x)$ is also invertible and

$$(L_a^f)^{-1}(x) = L_b^g(x) = L_{f(a)}^g(x).$$

Inverse Function Theorem

Example: Let $f(x) = x^3$ with $f^{-1}(y) = g(y) = y^{\frac{1}{3}}$. Let $a = 2$. Find

$$g'(f(a)) = g'(8).$$

Solution: We know that $f'(x) = 3x^2$, so by the Inverse Function Theorem:

$$\begin{aligned} g'(8) &= \frac{1}{f'(2)} \\ &= \frac{1}{12}. \end{aligned}$$

We also know that $g'(y) = \frac{1}{3}y^{-\frac{2}{3}}$, so

$$\begin{aligned} g'(8) &= \frac{1}{3} \cdot 8^{-\frac{2}{3}} \\ &= \frac{1}{12}. \end{aligned}$$

Inverse Function Theorem

Note:

Let $f(x) = x^3$ with $f^{-1}(y) = g(y) = y^{\frac{1}{3}}$.

Let $a = 0$, so $b = f(0) = 0$. We have

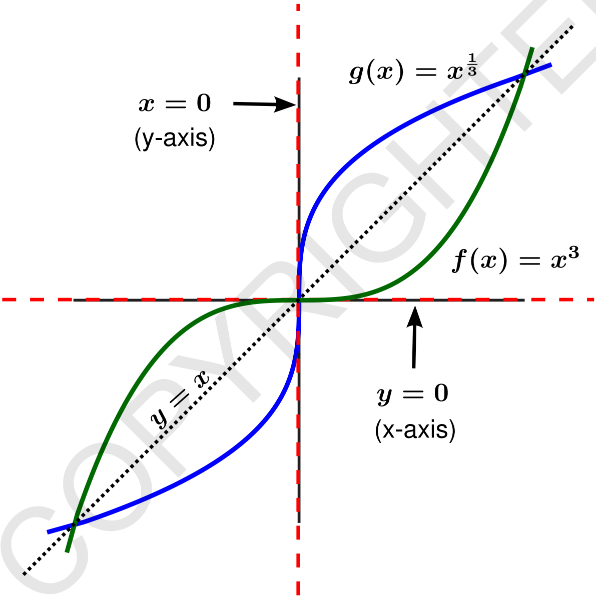
$$f'(0) = 3 \cdot 0^2 = 0$$

but

$$g'(y) = \frac{1}{3} y^{-\frac{2}{3}}$$

so $g(y)$ is not differentiable at $b = 0$.

Inverse Function Theorem



Derivative of $ln(x)$

Example: We know that $f(x) = \ln(x)$ is invertible with inverse $g(y) = e^y$. Since e^y is differentiable for every $y \in \mathbb{R}$ the Inverse Function Theorem tells us that $f(x) = \ln(x)$ is differentiable for all $x > 0$ and that

$$\begin{aligned} f'(x) &= \frac{1}{g'(f(x))} \\ &= \frac{1}{e^{f(x)}} \\ &= \frac{1}{e^{\ln(x)}} \\ &= \frac{1}{x}. \end{aligned}$$

Theorem: [Derivative of $\ln(x)$]

The function $f(x) = \ln(x)$ is differentiable at $x > 0$, and

$$f'(x) = \frac{1}{x}$$