Exponential Growth and Decay

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Exponential Growth and Decay

Remark: Let $Q(t)$ denote either the size of a bacterial population at time t, or the quantity of a radioactive substance at time t. Then in both cases it is known that the rate of change of the quantity $Q(t)$ is proportional to $Q(t)$ itself.

That is,

$$Q'(t) = kQ(t)$$

where $k \in \mathbb{R}$ is a fixed constant.

Problem: Find all functions $Q(t)$ satisfying the *differential equation*

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Solution: Let

$$Q_0(t) = e^{kt}.$$

Then

$$Q_0'(t) = ke^{kt} = kQ_0(t).$$

Assume that $Q'(t) = kQ(t)$ and let

$$H(t) = \frac{Q(t)}{e^{kt}}.$$

Then

$$\begin{aligned} H'(t) &= \frac{e^{kt}Q'(t) - ke^{kt}Q(t)}{e^{2kt}} \\ &= \frac{ke^{kt}Q(t) - ke^{kt}Q(t)}{e^{2kt}} \\ &= 0 \end{aligned}$$

so

$$C = H(t) = \frac{Q(t)}{e^{kt}} \Rightarrow Q(t) = Ce^{kt}.$$

Exponential Growth and Decay

Remarks:

1) Both the size of a bacterial population at time t, or the quantity of a radioactive substance at time t would then be of the form

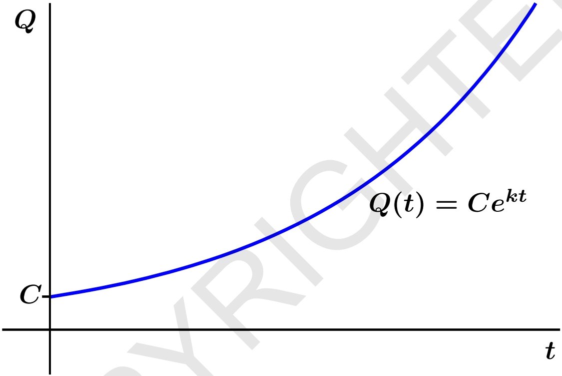
$$Q(t) = Ce^{kt}.$$

2) Since $Q(t) \geq 0$ this would mean that C is positive. Moreover, since

$$Q(0) = Ce^{k \cdot 0} = C$$

the constant C represents the initial population or the initial quantity present at time $t = 0$.

Exponential Growth and Decay



Remarks (continued):

3) In the case of the *growing* bacterial colony, the derivative $Q'(t) = kQ(t)$ must also be positive. This forces k to be greater than 0 (i.e., $k > 0$). Hence, the graph of the bacterial population looks like that of a typical exponential function.

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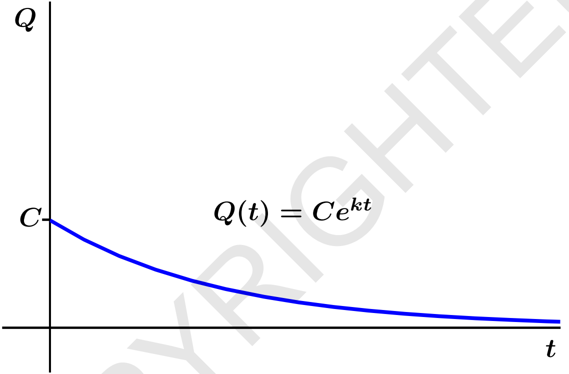

Remarks (continued):

4) In the case of radioactive *decay*, the quantity $Q(t)$ decreases with time. Consequently, k must be less than 0 (i.e., $k < 0$). This produces a graph that is typical of an exponential function with a base that is less than 1.

Exponential Growth and Decay

Example: A bacterial colony starts with a population of 1000. After 2 hours, the population is estimated to be 3500. What would you expect the population to be after 7 hours?

Solution: Let $P(t)$ denote the population of the bacterial colony t hours after the first estimate. Since the rate of growth of the population is proportional to the size of the population,

$$P(t) = Ce^{kt}.$$

Since C represents the initial population size, $C = 1000$ and

$$P(t) = 1000e^{kt}.$$

We also know that

$$3500 = P(2) = 1000e^{k \cdot 2}.$$

Solving for k gives us that

$$\frac{3500}{1000} = e^{2k} \Rightarrow k = \frac{\ln(3.5)}{2}.$$

Finally,

$$\begin{aligned} P(7) &= 1000e^{7\left(\frac{\ln(3.5)}{2}\right)} \\ &\approx 80212 \text{ bacteria.} \end{aligned}$$

Exponential Growth and Decay

Fact: The half-life of a radioactive isotope is the time it takes for a given quantity to decay to $\frac{1}{2}$ of its original mass.

The half-life of Uranium-238 (U-238) is 4.5×10^9 years. The biproduct of this decay is lead.

Example: During a lava flow, molten lead is separated from the rest of the lava so we can assume that a fresh lava flow will be free of lead. However, as U-238 decays in the flow, trace amounts of lead will appear. Use this to determine the age of a lava flow that contains 3 molecules of lead to every 5000 molecules of U-238.

Exponential Growth and Decay

Solution: The quantity $Q(t)$ of U-238 is decreasing at a rate proportional to the amount left at any given time, so

$$Q(t) = Ce^{kt}$$

where C represents the original quantity of the isotope.

We can find k by using the half-life of the isotope. In particular, we know that if $t_0 = 4.5 \times 10^9$, then

$$Ce^{kt_0} = Q(t_0) = \frac{C}{2}$$

so

$$e^{kt_0} = Q(t_0) = \frac{1}{2}.$$

Next take the natural logarithm of both sides of the equation to get

$$kt_0 = \ln\left(\frac{1}{2}\right)$$

and finally that

$$k = \frac{1}{t_0}\ln\left(\frac{1}{2}\right) = \frac{1}{4.5 \times 10^9}\ln\left(\frac{1}{2}\right).$$

Note that this shows k is actually independent of the initial quantity C.

Exponential Growth and Decay

Solution (continued): The data tells us that if we began with 5003 molecules of U-238, we would now have 5000. Therefore, if t_1 is the age of the lava flow, then $C = 5003$ and we have $Q(t_1) = 5000$. This means that

$$5000 = 5003e^{kt_1}$$

so

$$e^{kt_1} = \frac{5000}{5003}.$$

Finally,

$$\begin{aligned} t_1 &= \frac{\ln(\frac{5000}{5003})}{k} \\ &= (4.5 \times 10^9)\left(\frac{\ln(\frac{5000}{5003})}{\ln(\frac{1}{2})}\right) \\ &\approx 3.894108 \times 10^6 \text{ years.} \end{aligned}$$

Therefore, the lava flow is approximately 3.89 million years old.