The Derivative Function

Created by

Barbara Forrest and Brian Forrest

The Derivative at a Point

Recall:

Definition: [Derivative]

We say that the function $f(x)$ is differentiable at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

Equivalently:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

The Derivative Function

Definition: [Derivative Function]

We say that a function $f(x)$ is differentiable on an interval I if $f\,'(a)$ exists for every $a \in I$.

In this case, we define the derivative function on I, denoted by $f\,'$, where

$$f\,'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

That is, the value of the derivative function at x is simply the derivative of f at x for each $x \in I$.

Leibniz Notation

Leibniz Notation: Given a function $y = f(t)$, Leibniz wrote

$$\frac{dy}{dt} \quad \text{or} \quad \frac{df}{dt}$$

to represent the derivative of y (or equivalently, of f) with respect to t.

An alternate form of Leibniz's notation is to write

$$\frac{d}{dt}(f(t))$$

to indicate that $f(t)$ is to be differentiated with respect to the variable t.

The symbol

$$\frac{d}{dt}$$

is called a *differential operator*.

In Leibniz's notation, we denote $f'(a)$, the derivative at $t = a$, by

$$\frac{dy}{dt}\,|_a \quad \text{or} \quad \frac{df}{dt}\,|_a \;.$$

Higher Derivatives

Definition: [Higher Derivatives]

Let $f(x)$ be a differentiable function with derivative $f'(x)$. If $f'(x)$ is also differentiable, then its derivative

$$\frac{d}{dx}(f'(x))$$

is called the *second derivative* of $f(x)$ and it is usually denoted by

$$f''(x) \quad \text{or} \quad f^{(2)}(x) \quad \text{or} \quad \frac{d^2}{dx^2}(f(x)).$$

If $f''(x)$ is also differentiable, then its derivative is called the *third derivative* of $f(x)$ and it is denoted by

$$f'''(x) \text{ or by } f^{(3)}(x).$$

In general, for any $n \geq 1$,

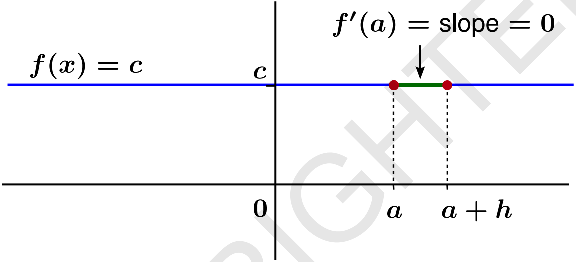
$$f^{(n+1)}(x) = \frac{d}{dx}(f^{(n)}(x))$$

and $f^{(n)}(x)$ is called the n-th derivative of $f(x)$.

Higher Derivatives

Note: $f''(x)$ impacts the geometry of the graph of $f(x)$. In particular, the larger the magnitude of $f''(x)$, the more *curved* the graph of $f(x)$.

Derivative of a Constant Function



Example: [Derivative of a Constant Function]

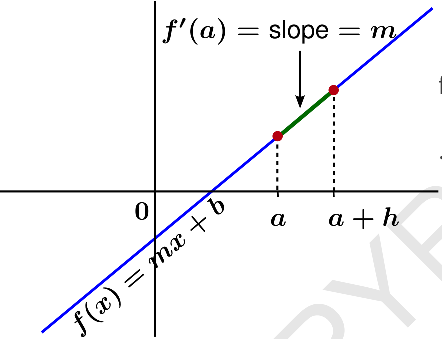
Assume that $f(x) = c$ for all $x \in \mathbb{R}$ and let $a \in \mathbb{R}$. Then

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= 0 \end{aligned}$$

Hence $f'(x) = 0$ for all $x \in \mathbb{R}$.

Derivative of a Linear Function

Example: [Derivative of a Linear Function]



Assume that

$$f(x) = mx + b$$

for all $x \in \mathbb{R}$. Then

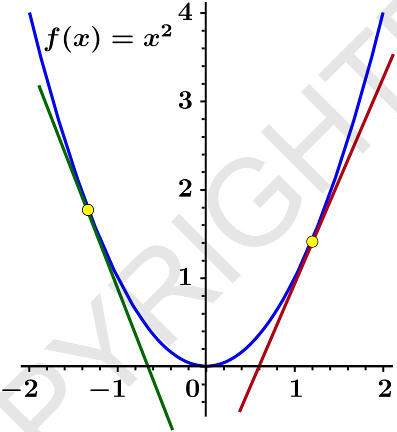
$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{(m(a+h) + b) - (ma + b)}{h} \\ &= \lim_{h \to 0} \frac{ma + mh + b - ma - b}{h} \\ &= \lim_{h \to 0} \frac{mh}{h} \\ &= m. \end{aligned}$$

Hence

$$f'(x) = m$$

for all $x \in \mathbb{R}$.

Derivative of $f(x) = x^2$

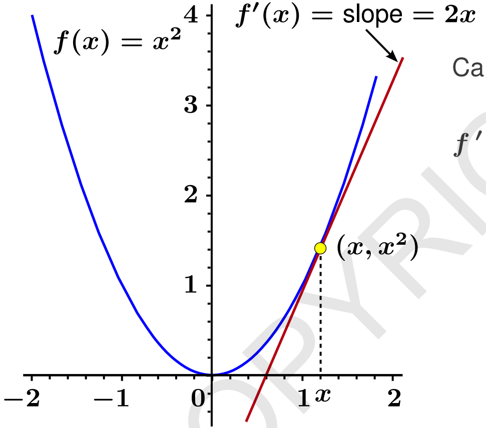


Example: Calculate the derivative of $f(x) = x^2$.

Note: Unlike the previous examples, the derivative appears to vary with the choice of x.

Derivative of $f(x) = x^2$

Example: [Derivative of a Quadratic Function]

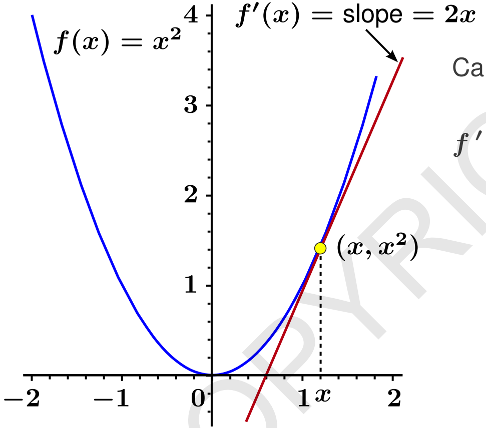


Calculate the derivative of $f(x) = x^2$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x.
\end{aligned}$$

Derivative of $f(x) = x^2$

Example: [Derivative of a Quadratic Function]



Calculate the derivative of $f(x) = x^2$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x.
\end{aligned}$$

Note: If $f(x) = x^2$, then $f'(x) = 2x$, so $f''(x) = 2$.