Derivatives and Continuity

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Derivatives and Continuity

Definition: [Derivative]

We say that the function $f(t)$ is differentiable at $t = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

Equivalently:

$$f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$$

if this limit exists.

Derivatives and Continuity

Question: Is there a relationship between differentiability and continuity?

Key Observation: Assume that $f(t)$ is differentiable at $t = a$. Then

$$f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$$

so

$$0 = f'(a) \cdot \lim_{t \to a}(t - a) = \lim_{t \to a} f(t) - f(a).$$

Hence

$$\lim_{t \to a} f(t) = f(a)$$

and $f(t)$ is continuous at $t = a$.

Derivatives and Continuity

Theorem: [Differentiation Implies Continuity]

Assume that $f(t)$ is differentiable at $t = a$. Then $f(t)$ is continuous at $t = a$.

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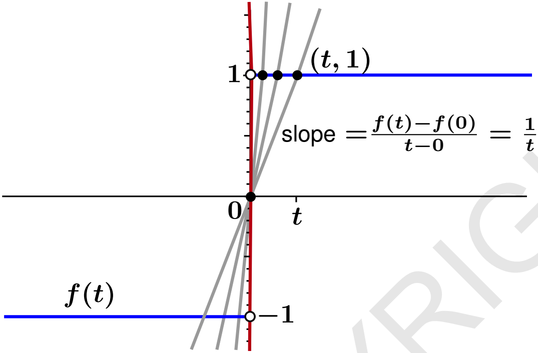

Example: Consider

$$f(t) = \begin{cases} \frac{|t|}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Then we know that

$$f(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}.$$

Then

$$\lim_{t\to 0^+} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0^+} \frac{1}{t} = \infty$$

since $f(0) = 0$ and $f(t) = 1$ if $t > 0$.

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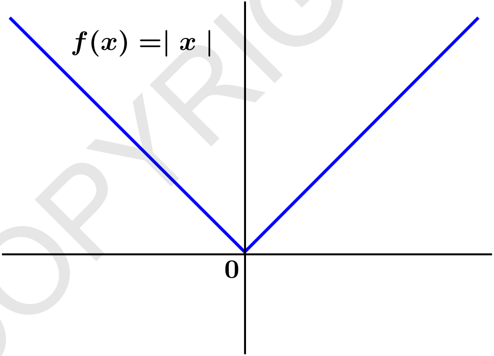
Example: Let $f(x) = \mid x \mid$ and let $a = 0$. Then $f(x)$ is continuous at 0. However, since $f(0) = 0$, we get that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\mid x \mid}{x}.$$



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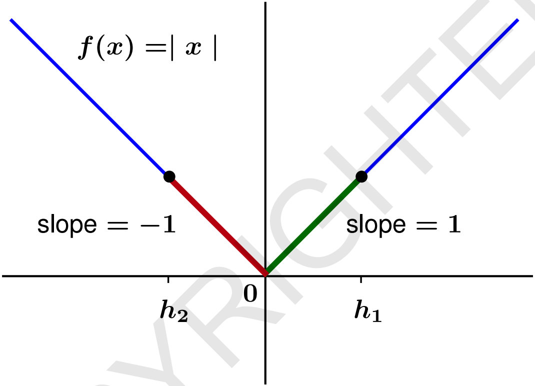

Example (continued):

If $h_1 > 0$, then the slope of the secant line through $(0, f(0)) = (0, 0)$ and $(h_1, f(h_1)) = (h_1, h_1)$ is 1.

If $h_2 < 0$, then the slope of the secant line through $(0, f(0)) = (0, 0)$ and $(h_2, f(h_2)) = (h_2, -h_2)$ is -1.

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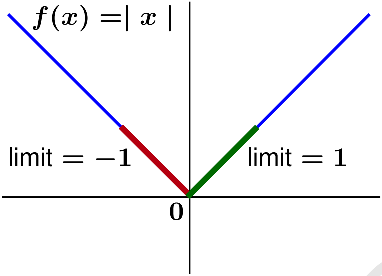


Example (continued): Therefore

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

while

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

Hence $f(x) = |x|$ is continuous, but not differentiable at $x = 0$.

Derivatives and Continuity

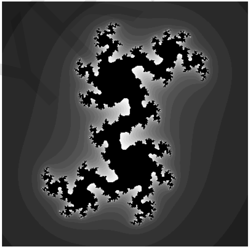
Historically Important Question: Does there exist a function $f(x)$ that is continuous at each $x \in \mathbb{R}$, but not differentiable at even one point?

Answer: Yes! [Karl Weierstrass (1872)]

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n x)$$

Note:

1) The most famous application of these types of functions are *fractals*.



2) It can be shown that *most* continuous functions are nowhere differentiable.