Derivatives

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Average Rate of Change

Definition: [Average Rate of Change]
Given a function $f(t)$, we can define the average rate of change of $f(t)$ as t goes from t_0 to t_1 to be the ratio

$$\frac{f(t_1) - f(t_0)}{t_1 - t_0}.$$

Note: If we fix a point t_0 and let $h = t_1 - t_0$ be small, then

$$\frac{f(t_0 + h) - f(t_0)}{h}$$

is an estimate of the *instantaneous rate of change of* $f(t)$ at t_0.

Definition: [Instantaneous Rate of Change]
Given a function $f(t)$, we can define the instantaneous rate of change of $f(t)$ at t_0 to be

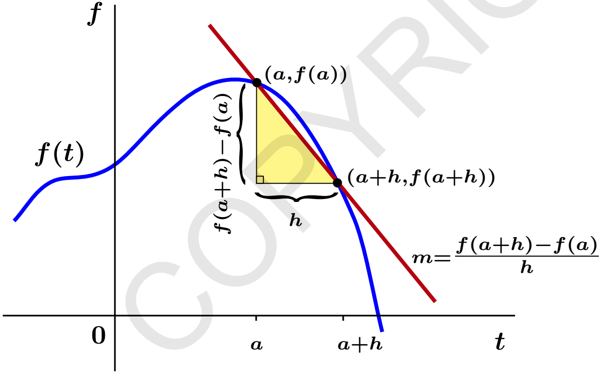
$$\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

The Derivative

Definition: [Newton Quotient]
Given a function $f(t)$, a point $t = a$ and $h \neq 0$, the ratio

$$\frac{f(a+h) - f(a)}{h}$$

is called a *Newton Quotient* for $f(t)$ centered at $t = a$.



Note: Geometrically, the Newton Quotient represents the slope of the secant line to the graph of $f(t)$ through the points $(a, f(a))$ and $(a+h, f(a+h))$.

The Derivative

Definition: [The Derivative at $t = a$]
We say that the function $f(t)$ is differentiable at $t = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

In this case, we write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and $f'(a)$ is the derivative of $f(t)$ at $t = a$.

Note: If $t = a + h$, then as $h \to 0$ we have $t \to a$. Furthermore, since $h = t - a$, we get

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{t \to a} \frac{f(t) - f(a)}{t - a} \end{aligned}$$

provided the limits exist.

Derivative as Instantaneous Rate of Change

Observation: If $y = f(t)$, $\triangle y = f(t) - f(a)$ and $\triangle t = t - a$, then the Newton Quotient

$$\frac{\triangle y}{\triangle t} = \frac{f(t) - f(a)}{t - a}$$

is the *average* change in y.

Therefore,

$$f'(a) = \lim_{\triangle t \to 0} \frac{\triangle y}{\triangle t}$$

is the limit of average rates of change over smaller and smaller intervals and so this represents the *instantaneous rate of change of y with respect to t.*

Derivative as Instantaneous Rate of Change

Example:

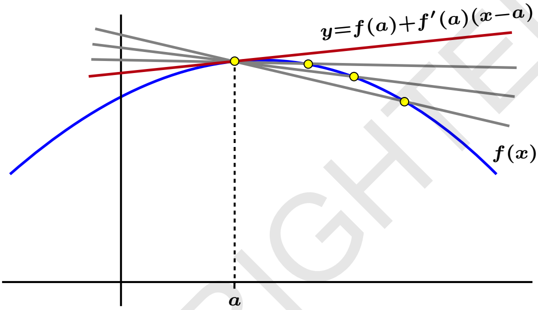
Let $s = s(t)$ represent the displacement of an object. Then

$$\begin{aligned} s'(t_0) &= \lim_{\triangle t \to 0} \frac{\triangle s}{\triangle t} \\ &= \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} \\ &= \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h} \\ &= v(t_0) \end{aligned}$$

where $v(t_0)$ is the instantaneous velocity of the object at time t_0.

Fact: ***Velocity is the derivative of displacement.***

The Tangent Line



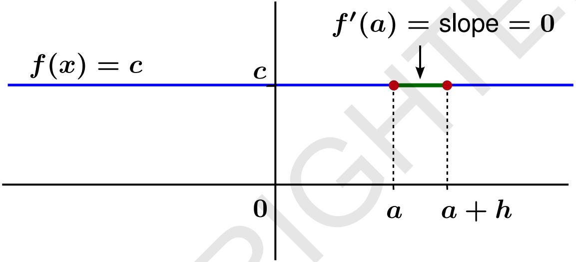
Question: Does there exist a geometric interpretation of the derivative?

Observation: If $f(x)$ is differentiable at $x = a$, the slopes of the secant lines through $(a, f(a))$ and $(a + h, f(a + h))$ converge to $f\,'(a)$ as $h \to 0$.

Definition: [Tangent Line]
The *tangent line to the graph of* $f(x)$ *at* $x = a$ is the line passing through $(a, f(a))$ with slope equal to $f'(a)$. That is,

$$y = f(a) + f\,'(a)(x - a).$$

Derivative of a Constant Function



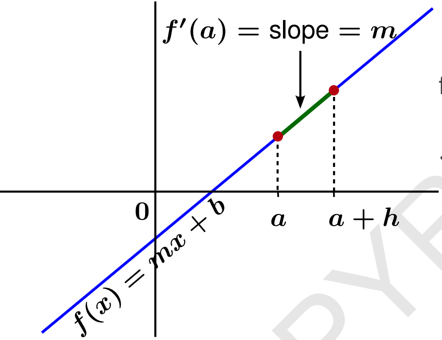
Example: [Derivative of a Constant Function]

Assume that $f(x) = c$ for all $x \in \mathbb{R}$ and let $a \in \mathbb{R}$. Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 0$$

Derivative of a Linear Function

Example: [Derivative of a Linear Function]



Assume that

$$f(x) = mx + b$$

for all $x \in \mathbb{R}$. Then

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{(m(a+h)+b) - (ma+b)}{h} \\
&= \lim_{h\to 0} \frac{ma + mh + b - ma - b}{h} \\
&= \lim_{h\to 0} \frac{mh}{h} \\
&= m.
\end{aligned}$$