Derivatives of Sine and Cosine

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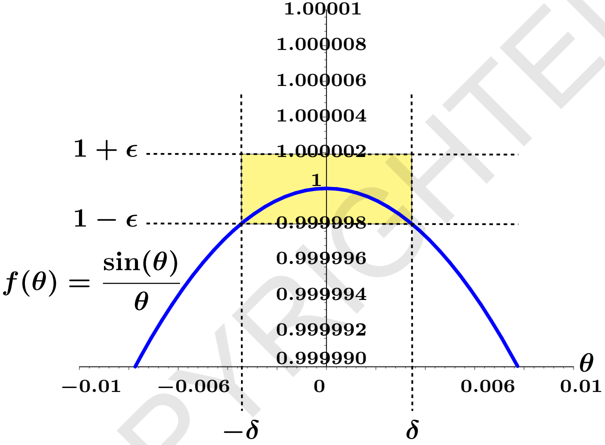
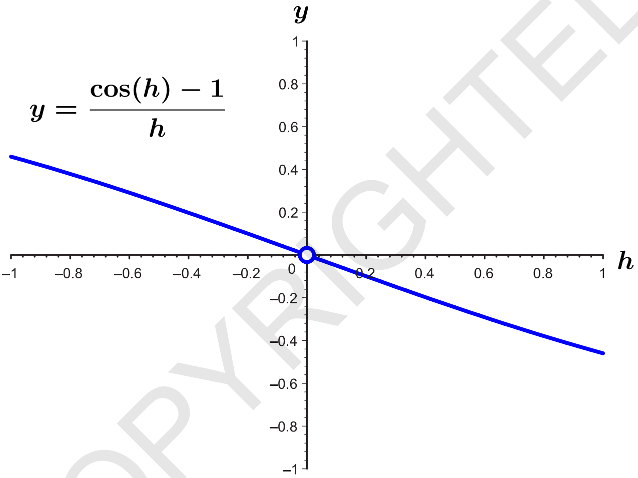
Derivative of $\sin(x)$



Theorem: [Fundamental Trig Limit] (FTL)

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

Derivative of $\sin(x)$



Theorem:

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0.$$

Derivative of $\sin(x)$

Theorem:

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0.$$

Proof: We have

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos(h)-1}{h} &= \lim_{h\to 0} \left(\frac{\cos(h)-1}{h}\right)\left(\frac{\cos(h)+1}{\cos(h)+1}\right) \\
&= \lim_{h\to 0} \frac{\cos^2(h)-1}{h(\cos(h)+1)} \\
&= \lim_{h\to 0} \frac{-\sin^2(h)}{h(\cos(h)+1)} \\
&= \lim_{h\to 0} \frac{\sin(h)}{h} \cdot \lim_{h\to 0} \frac{-\sin(h)}{(\cos(h)+1)} \\
&= 1 \cdot 0 \\
&= 0
\end{aligned}$$

since $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$, $\lim_{h\to 0} (\sin(h)) = 0$ and $\lim_{h\to 0} (\cos(h)+1) = 2$.

Derivative of $\sin(x)$

Problem: Find $\frac{d}{dx}(\sin(x))$.

Solution:

$$\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(\sin(x)\cos(h)+\cos(x)\sin(h))-\sin(x)}{h} \\
&= \lim_{h\to 0} \left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 \\
&= \cos(x).
\end{aligned}$$

Derivative of $\sin(x)$

Theorem: [The Derivative of $\sin(x)$]

Assume that $f(x) = \sin(x)$. Then

$$f'(x) = \cos(x)$$

for all $x \in \mathbb{R}$.

Derivative of $\cos(x)$

Theorem: [The Derivative of $\cos(x)$]

Assume that $f(x) = \cos(x)$. Then

$$f'(x) = -\sin(x)$$

for all $x \in \mathbb{R}$.

Proof: We have

$$\begin{aligned}
&\lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&\qquad = \lim_{h \to 0} \frac{(\cos(x)\cos(h) - \sin(x)\sin(h)) - \cos(x)}{h} \\
&\qquad = \lim_{h \to 0} \left(\cos(x) \frac{\cos(h) - 1}{h} - \sin(x) \frac{\sin(h)}{h} \right) \\
&\qquad = \cos(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h} \\
&\qquad = \cos(x) \cdot 0 - \sin(x) \cdot 1 \\
&\qquad = -\sin(x).
\end{aligned}$$