Derivatives of Inverse Trigonometric Functions

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Inverse Function Theorem

Theorem: [Inverse Function Theorem (IFT)]

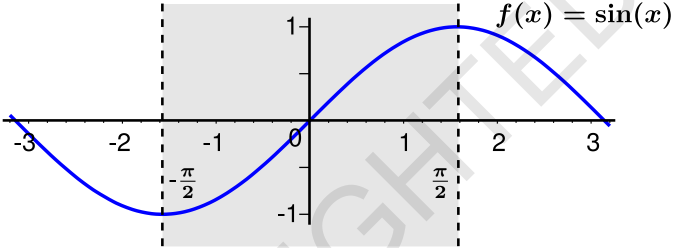
Assume that $f(x)$ is invertible on $[c, d]$ with inverse $g(y)$, and $f(x)$ is differentiable at $a \in (c, d)$. If $f'(a) \neq 0$, then $g(y)$ is differentiable at $b = f(a)$, and

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

Moreover, $L_a^f(x)$ is also invertible and

$$(L_a^f)^{-1}(x) = L_b^g(y) = L_{f(a)}^g(y).$$

arcsin(x)



Example: $\arcsin(x)$

Observe that $\sin(x)$ is not 1-1. However, if we restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\sin(x)$ is 1-1 on this interval.

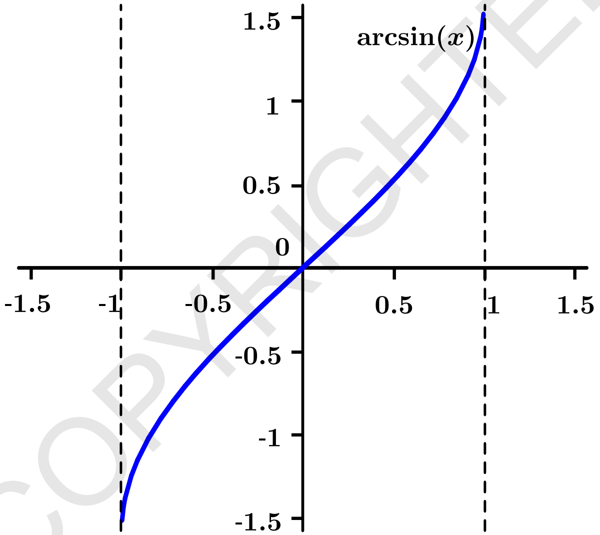
Definition [arcsin]:

Define $\arcsin(y) : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ by

$$\arcsin(y) = x \text{ if and only if } \sin(x) = y$$

for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

arcsin(x)



arcsin(x)

Problem: Find $\frac{d}{dx}\arcsin(x)$.

By the IFT

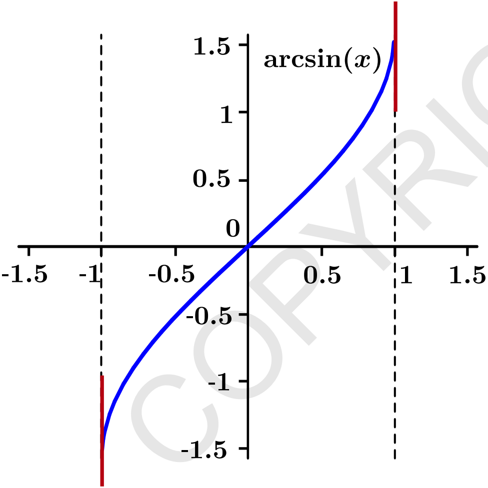
$$\begin{aligned}
\frac{d}{dx}\arcsin(x) &= \frac{1}{\frac{d}{dy}\sin(y)} \\
&= \frac{1}{\cos(y)} \\
&= \frac{1}{\sqrt{1-\sin^2(y)}} \\
&= \frac{1}{\sqrt{1-\sin^2(\arcsin(x))}} \\
&= \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

arcsin(x)

We have

$$h(x) = \frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

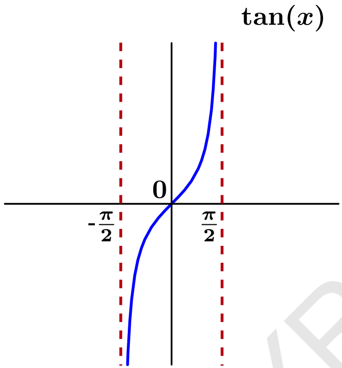
for $x \in (-1, 1)$.



Note:

1. $h(x) = \frac{1}{\sqrt{1-x^2}} > 0.$
2. $h(0) = \frac{1}{\sqrt{1-0^2}} = 1.$
3. $\lim_{x \to -1^+} \frac{1}{\sqrt{1-x^2}} = \infty$.
4. $\lim_{x \to 1^-} \frac{1}{\sqrt{1-x^2}} = \infty.$

arctan(x)



Example: $\arctan(x)$

Observe that $\tan(x)$ is not 1-1.

However, if we restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$, then $\tan(x)$ is 1-1 on this interval.

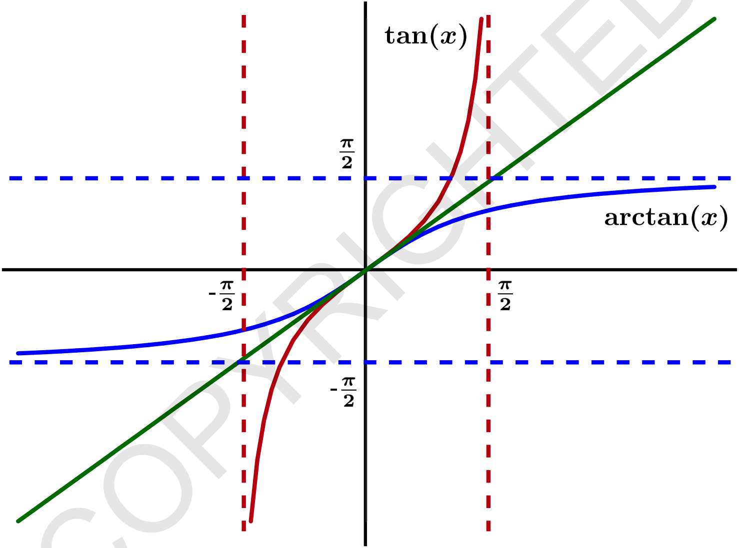
Definition [arctan]:

Define $\arctan(y) : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ by

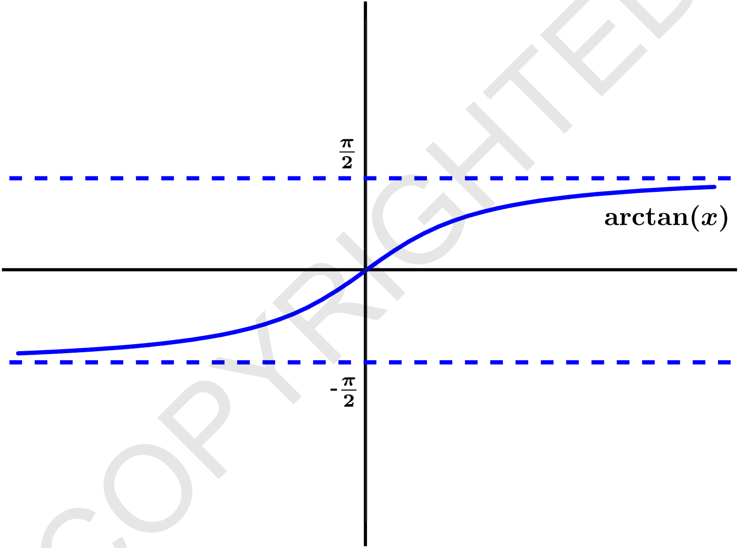
$$\arctan(y) = x \text{ if and only if } \tan(x) = y$$

for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

arctan(x)



arctan(x)



arctan(x)

Problem: Find $\frac{d}{dx}\arctan(x)$.

By the IFT

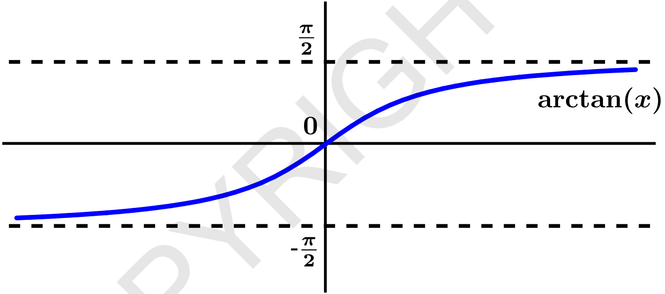
$$\begin{aligned}\frac{d}{dx}\arctan(x) &= \frac{1}{\frac{d}{dy}\tan(y)} \\ &= \frac{1}{\sec^2(y)} \\ &= \frac{1}{1+\tan^2(y)} \\ &= \frac{1}{1+\tan^2(\arctan(x))} \\ &= \frac{1}{1+x^2}.\end{aligned}$$

arctan(x)

We have

$$h(x) = \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

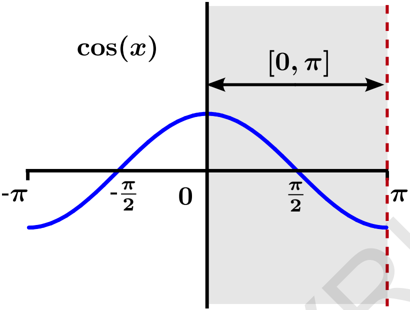
for $x \in \mathbb{R}$.



Note:

1. $h(x) = \frac{1}{1+x^2} > 0.$
2. $h(0) = \frac{1}{1+0^2} = 1.$
3. $\lim_{x \to -\infty} \frac{1}{1+x^2} = 0 = \lim_{x \to \infty} \frac{1}{1+x^2}.$

arccos(x)



Example: $\arccos(x)$

Observe that $\cos(x)$ is not 1-1.

However, if we restrict the domain to $[0, \pi]$, then $\cos(x)$ is 1-1 on this interval.

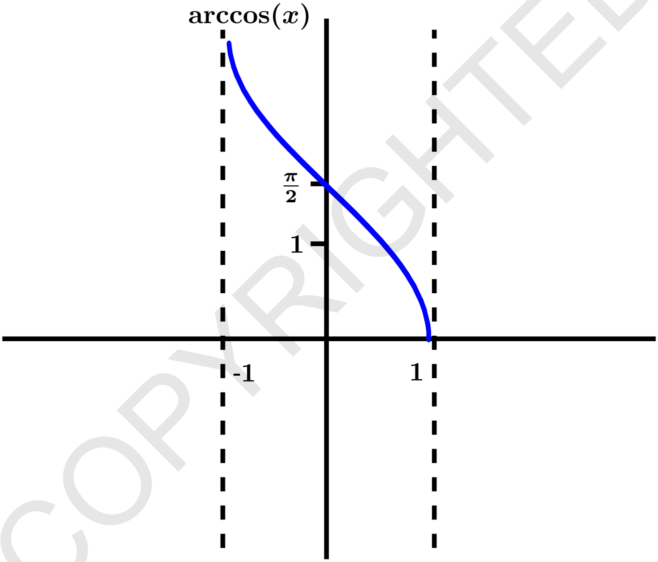
Definition [arccos]:

Define $\arccos(y) : [-1, 1] \to [0, \pi]$ by

$$\arccos(y) = x \text{ if and only if } \cos(x) = y$$

for $x \in [0, \pi]$.

$\arccos(x)$



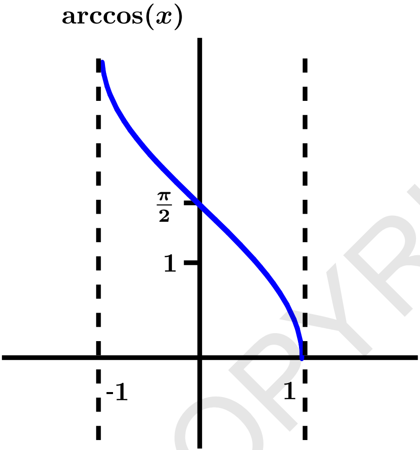
$\arccos(x)$

Problem: Find $\frac{d}{dx}\arccos(x)$.

By the IFT

$$\begin{aligned}
\frac{d}{dx}\arccos(x) &= \frac{1}{\frac{d}{dy}\cos(y)} \\
&= \frac{1}{-\sin(y)} \\
&= \frac{-1}{\sqrt{1-\cos^2(y)}} \\
&= \frac{-1}{\sqrt{1-\cos^2(\arccos(x))}} \\
&= \frac{-1}{\sqrt{1-x^2}}.
\end{aligned}$$

arccos(x)



We have

$$h(x) = \frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

for $x \in (-1, 1)$.

Note:

1. $h(x) = \frac{-1}{\sqrt{1-x^2}} < 0.$
2. $h(0) = \frac{-1}{\sqrt{1-0^2}} = -1.$
3. $\lim_{x\to -1^+} \frac{-1}{\sqrt{1-x^2}} = -\infty.$
4. $\lim_{x\to 1^-} \frac{-1}{\sqrt{1-x^2}} = -\infty.$