The Chain Rule

Created by

Barbara Forrest and Brian Forrest

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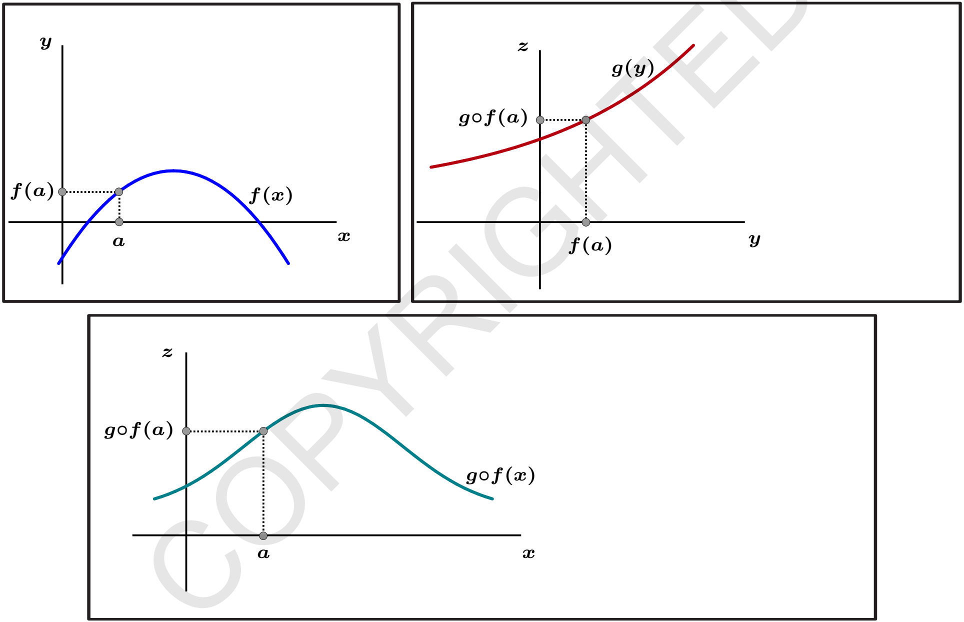
Chain Rule



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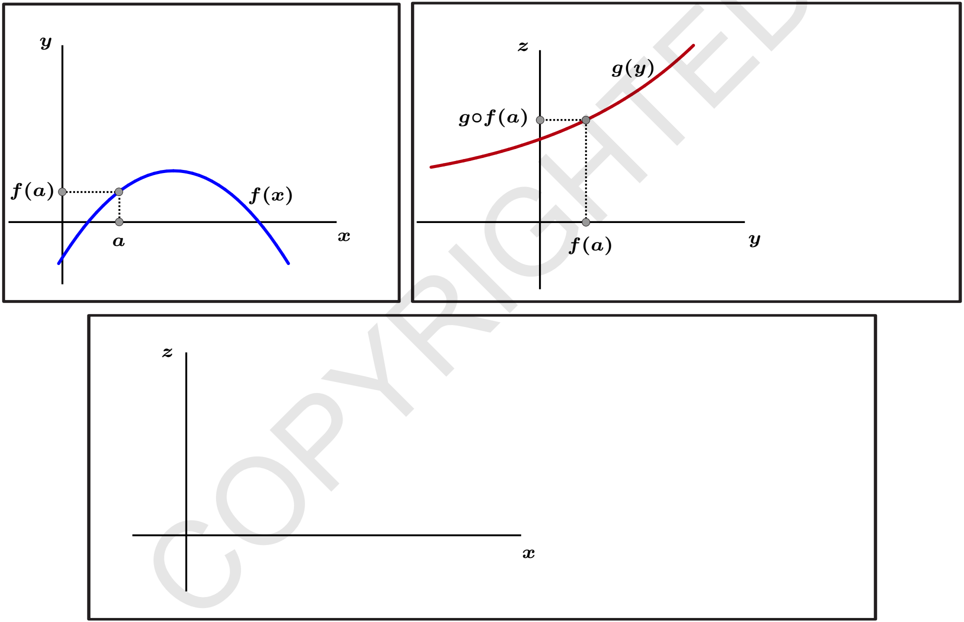

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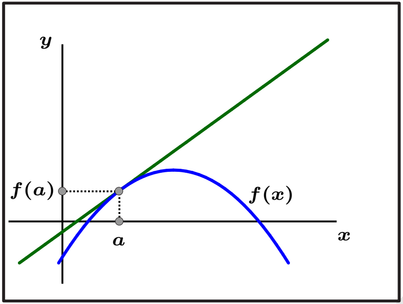

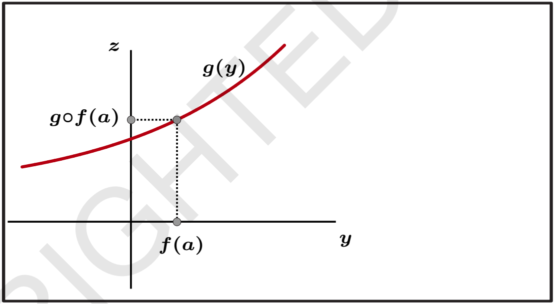

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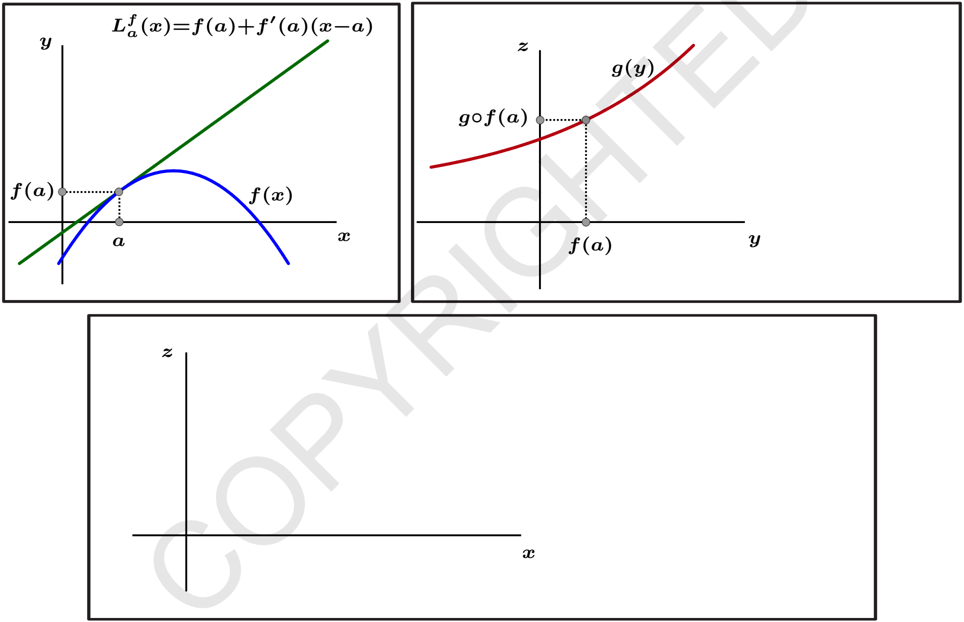

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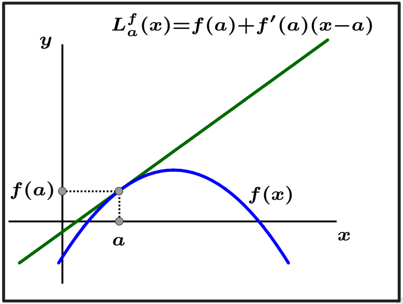

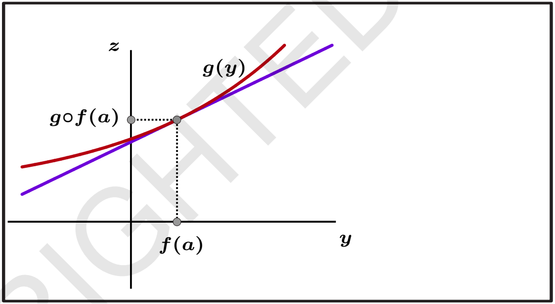

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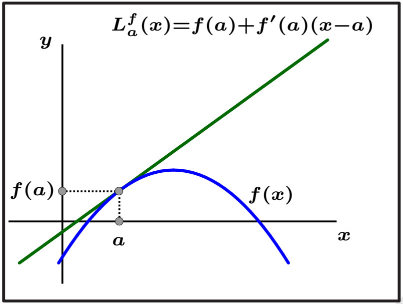

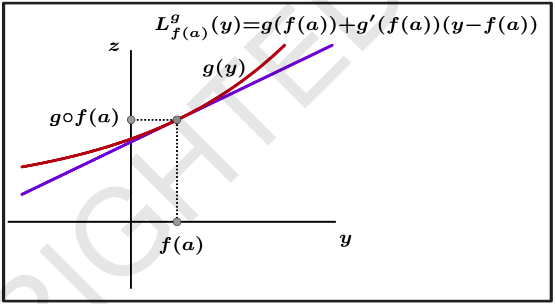

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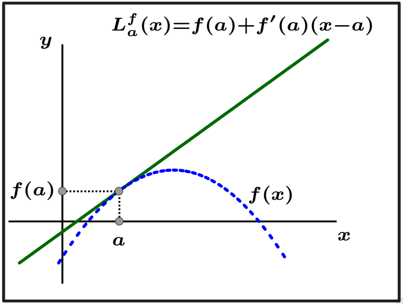

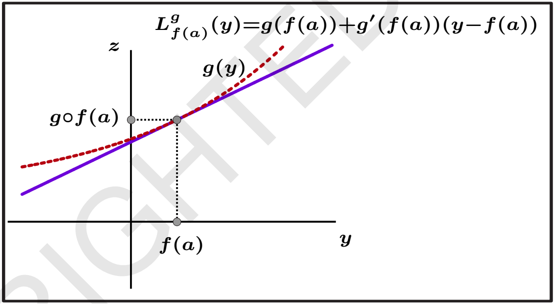

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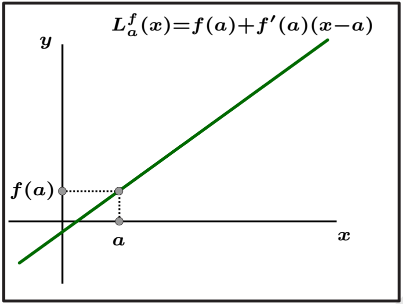

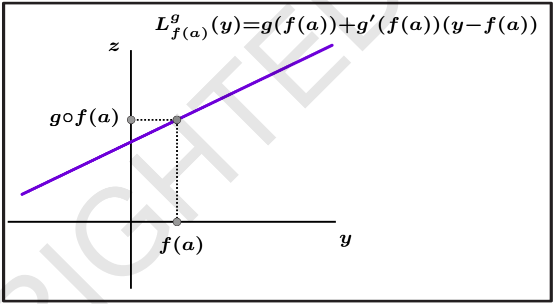

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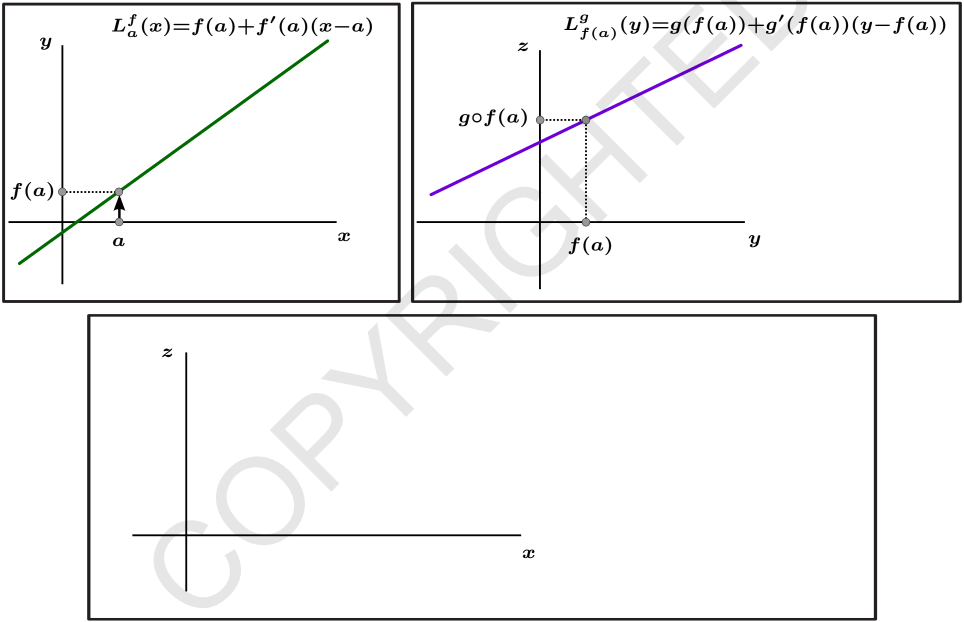

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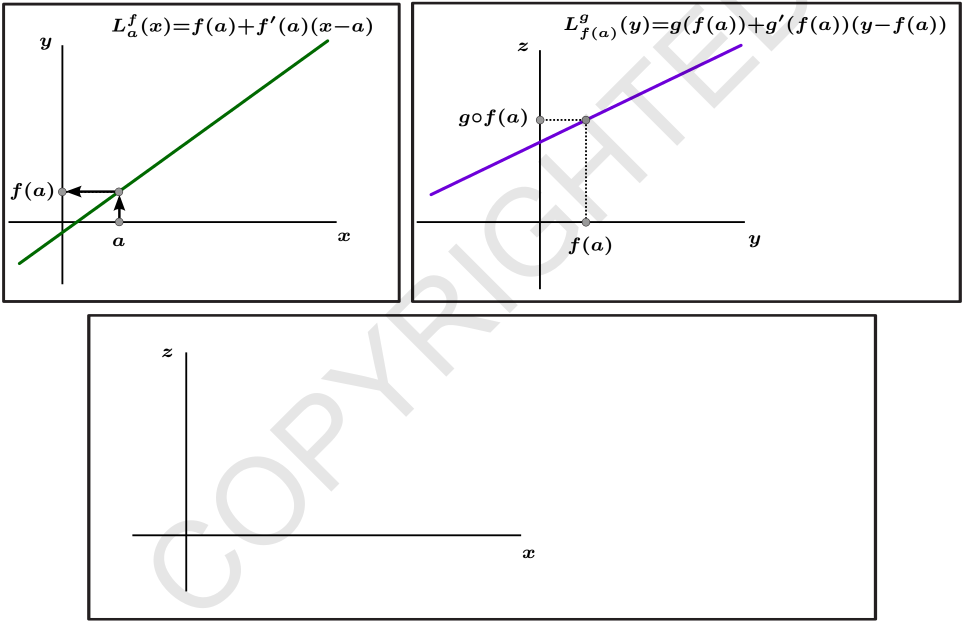

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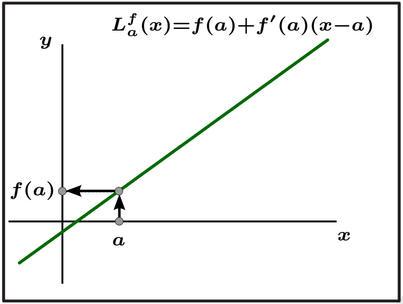

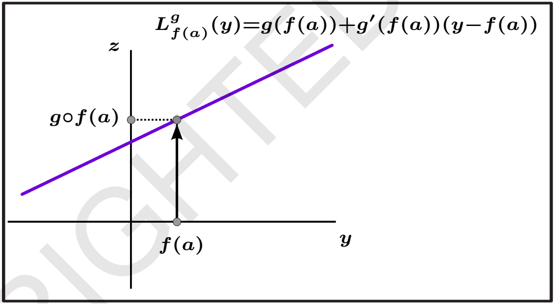

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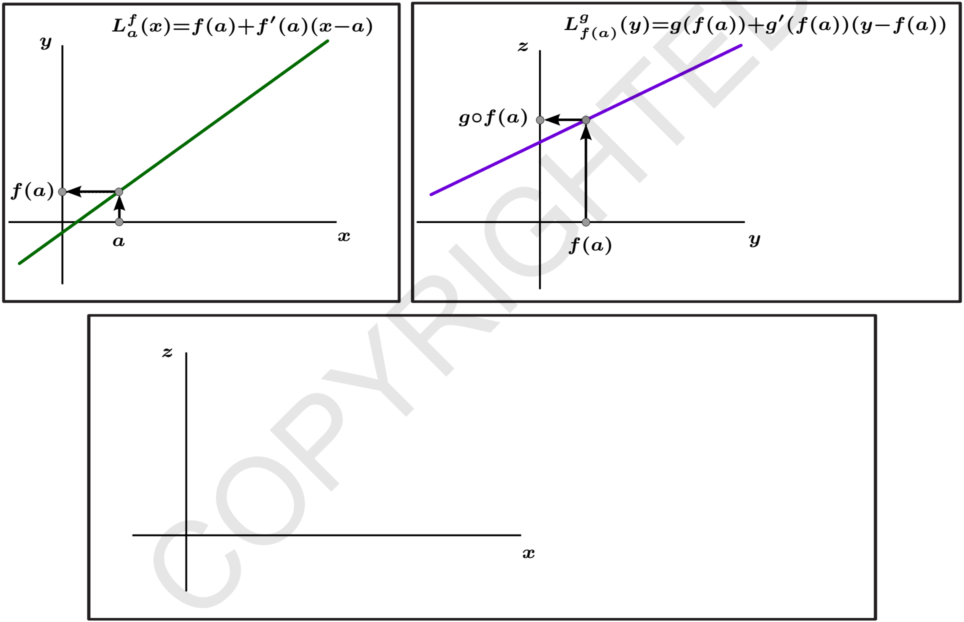

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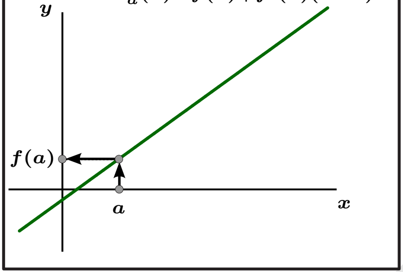
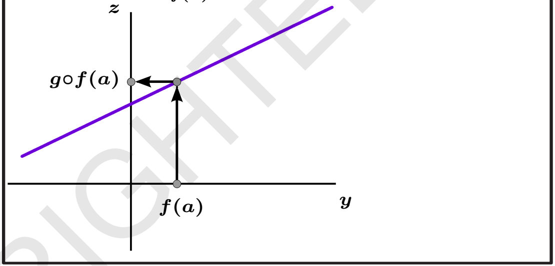
Chain Rule

$$L_a^f(x) = f(a) + f'(a)(x-a)$$



$$L_{f(a)}^g(y) = g(f(a)) + g'(f(a))(y - f(a))$$



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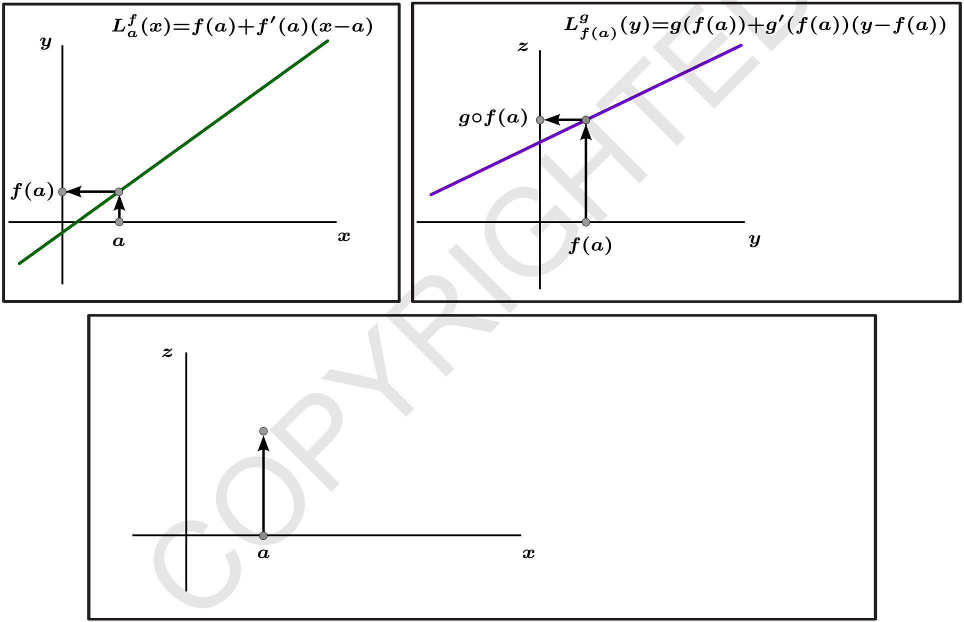

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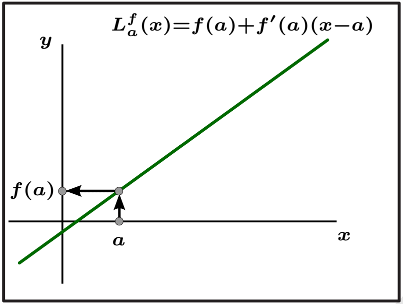

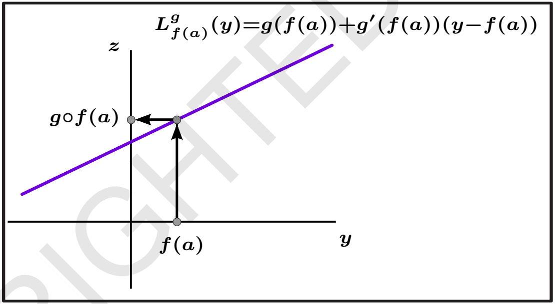

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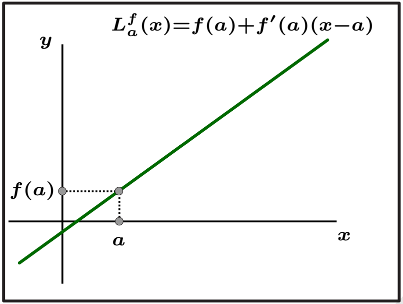

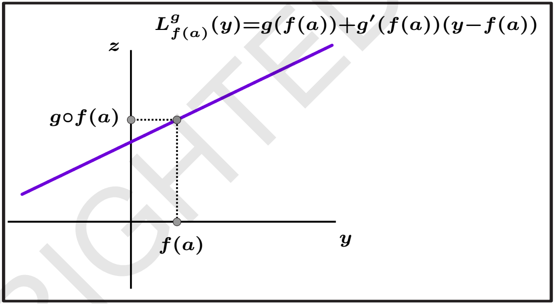

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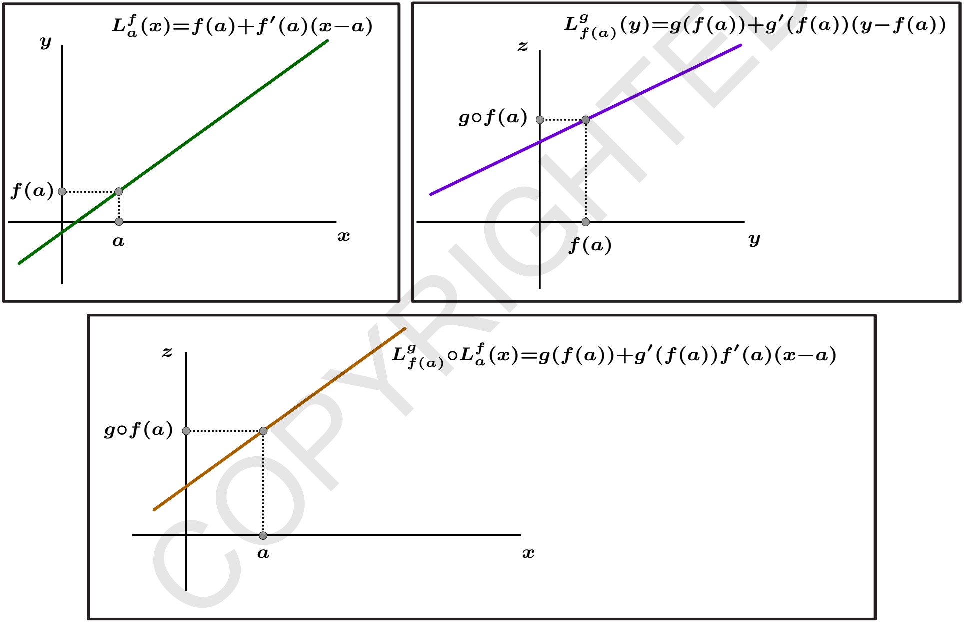

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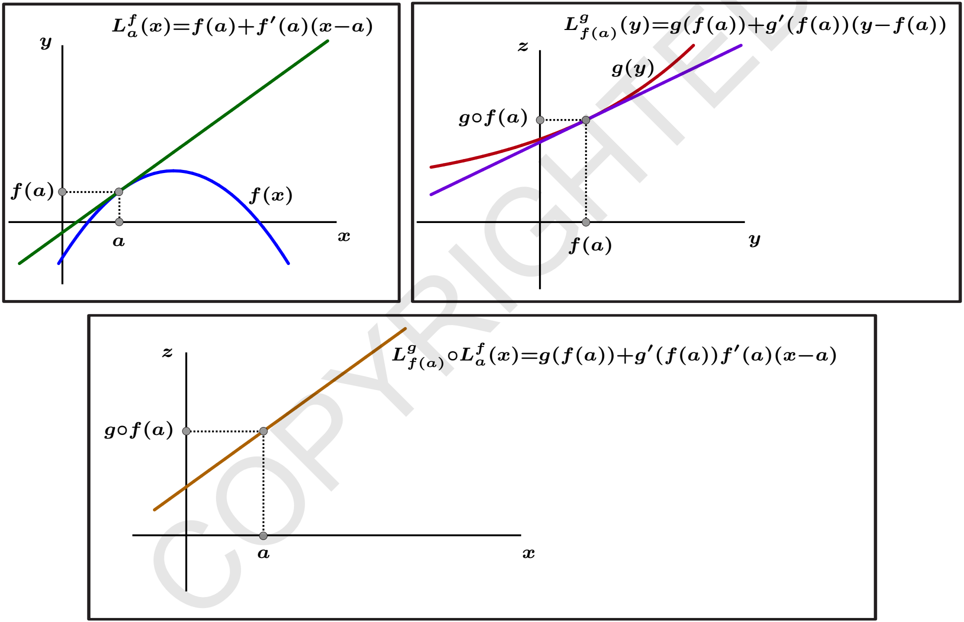

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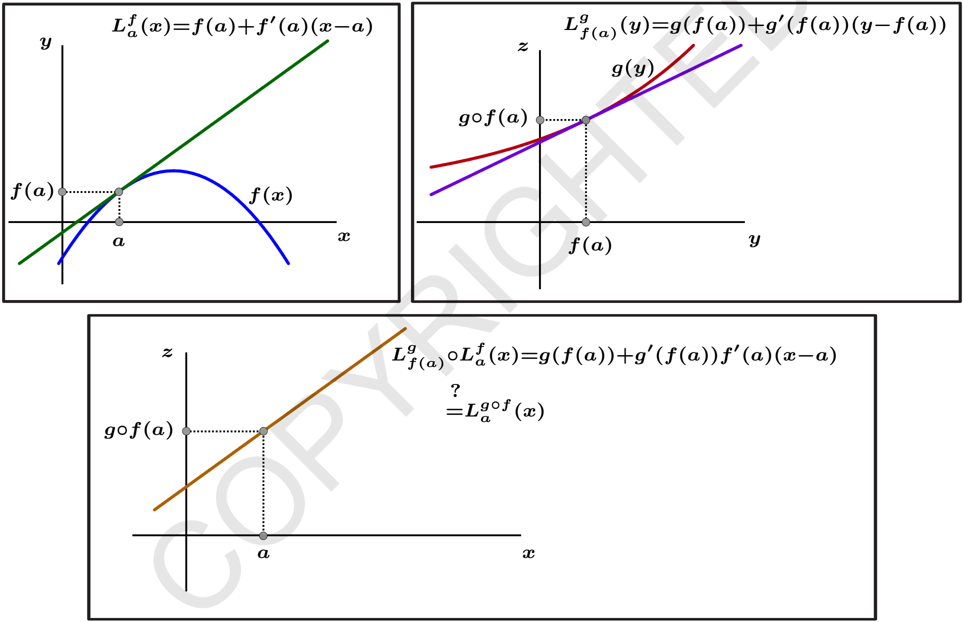

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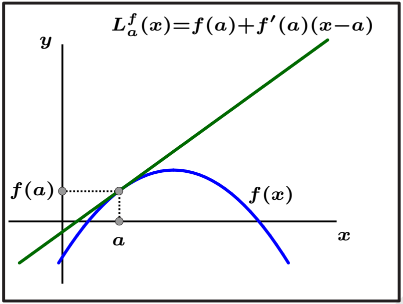

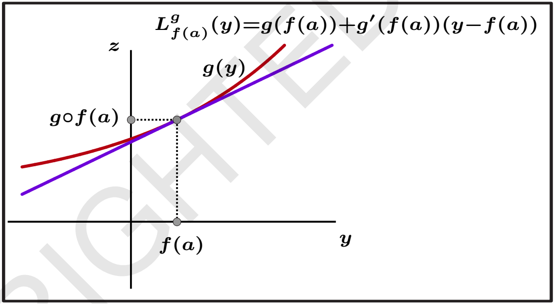

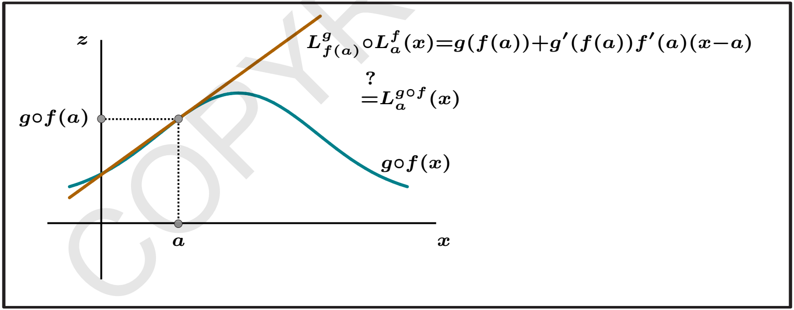

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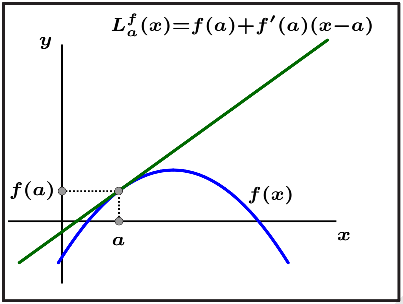

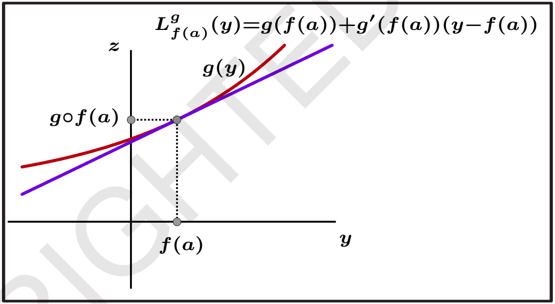

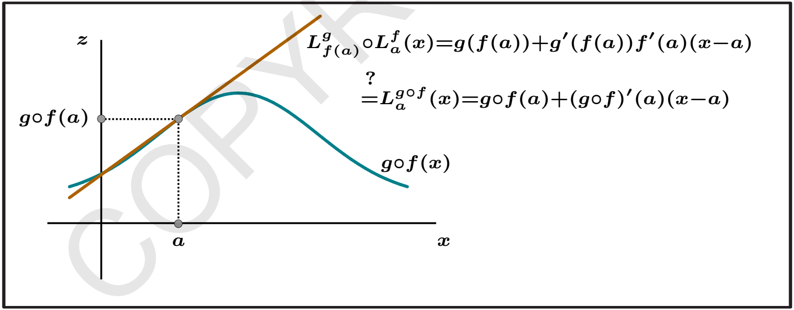

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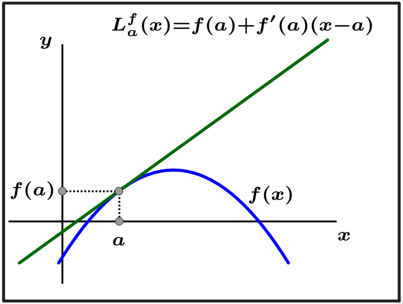

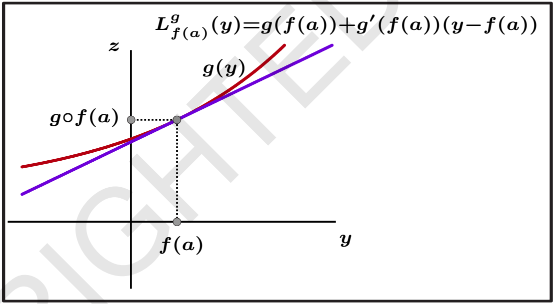

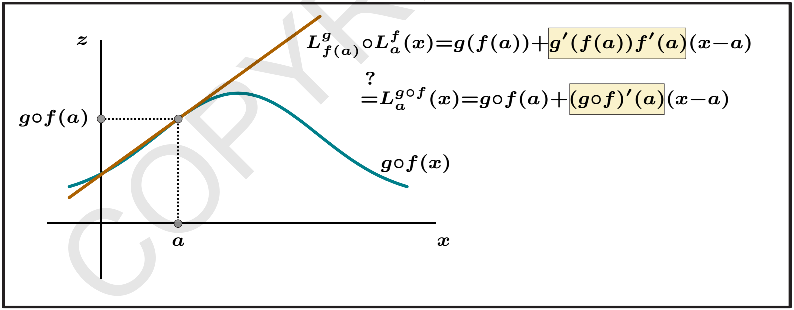

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Chain Rule

Theorem: [Chain Rule]

Assume that $f(x)$ is differentiable at $x = a$ and $g(y)$ is differentiable at $y = f(a)$. Then $h(x) = g \circ f(x) = g(f(x))$ is differentiable at $x = a$ and

$$h'(a) = g'(f(a))f'(a).$$

In particular,

$$L_a^h(x) = L_{f(a)}^g \circ L_a^f(x).$$

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Chain Rule

Problem: Assume that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both differentiable. Let $h(x) = g \circ f(x)$.

The equation of the tangent line to $f(x)$ through the point $(1, 3)$ is

$$y = 3 + 5(x - 1)$$

and the equation of the tangent line to the graph of $h(x)$ through $(1, 4)$ is

$$z = 4 - 2(x - 1).$$

Find the equation of the tangent line to the graph of $z = g(y)$ through the point $(3, 4)$.

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Chain Rule

Solution: The fact that the tangent line to $f(x)$ through the point $(1,3)$ is

$$y = 3 + 5(x-1) = L_1^f(x) = f(1) + f'(1)(x-1)$$

tells us that $f(1) = 3$ and that $f'(1) = 5$.

Similarly, the fact that the tangent line to the graph of $h(x)$ through $(1,4)$ is given by

$$z = 4 - 2(x-1) = L_1^h(x) = h(1) + h'(1)(x-1)$$

tells us that $h(1) = 4$ and $h'(1) = -2$.

Now $4 = h(1) = g(f(1)) = g(3)$ and from the Chain Rule, we get that

$$-2 = h'(1) = g'(f(1))f'(1) = g'(3)(5) \Rightarrow g'(3) = -\frac{2}{5}.$$

It follows that the equation of the tangent line to the graph of $z = g(y)$ through the point $(3,4)$ is given by

$$z = 4 - \frac{2}{5}(y-3).$$

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The Derivative of a^x

Example: If $h(x) = a^x$, find $h'(x)$.

Solution: Note that

$$a^x = e^{\ln(a)\cdot x}\ .$$

If $g(u) = e^u$ and $f(x) = \ln(a) \cdot x$, then

$$a^x = h(x) = g \circ f(x).$$

By the Chain Rule

$$\begin{aligned} h'(x) &= g'(f(x)) \cdot f'(x) \\ &= e^{\ln(a)\cdot x} \cdot \ln(a) \\ &= \ln(a) \cdot a^x. \end{aligned}$$

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