Arithmetic Rules for Differentiation

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Arithmetic Rules for Differentiation

Theorem: [The Arithmetic Rules for Differentiation]

Assume that $f(x)$ and $g(x)$ are both differentiable at $x = a$.

1) **The Constant Multiple Rule:**
   Let $h(x) = cf(x)$. Then $h(x)$ is differentiable at $x = a$ and

$$h'(a) = c \cdot f'(a).$$

2) **The Sum Rule:**
   Let $h(x) = f(x) + g(x)$. Then $h(x)$ is differentiable at $x = a$ and

$$h'(a) = f'(a) + g'(a).$$

3) **The Product Rule:**
   Let $h(x) = f(x)g(x)$. Then $h(x)$ is differentiable at $x = a$ and

$$h'(a) = f'(a)g(a) + f(a)g'(a).$$

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Theorem: [The Arithmetic Rules for Differentiation (continued)]

Assume that $f(x)$ and $g(x)$ are both differentiable at $x = a$.

4) **The Reciprocal Rule:**
   Let $h(x) = \frac{1}{f(x)}$. If $f(a) \neq 0$, then $h(x)$ is differentiable at $x = a$ and
$$h'(a) = \frac{-f'(a)}{(f(a))^2}.$$

5) **The Quotient Rule:**
   Let $h(x) = \frac{f(x)}{g(x)}$. If $g(a) \neq 0$, then $h(x)$ is differentiable at $x = a$ and
$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

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1) **Proof of the Constant Multiple Rule:**

Assume that $c \in \mathbb{R}$ and that $f(x)$ is differentiable at $x = a$. Then

$$\begin{aligned}
(cf)'(a) &= \lim_{h\to 0} \frac{(cf)(a+h) - (cf)(a)}{h} \\
&= \lim_{h\to 0} \frac{c \cdot f(a+h) - c \cdot f(a)}{h} \\
&= c \cdot (\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}) \\
&= c \cdot f'(a).
\end{aligned}$$

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2) **Proof of the Sum Rule:**

Assume that $f(x)$ and $g(x)$ are differentiable at $x = a$. Then

$$
\begin{aligned}
(f+g)'(a) &= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h) + g(a+h) - f(a) - g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\
&= f'(a) + g'(a).
\end{aligned}
$$

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3) **Proof of the Product Rule:**

Observe: We have

$$(fg)(a+h)-(fg)(a) = [f(a+h)g(a+h)-f(a+h)g(a)] + [f(a+h)g(a)-f(a)g(a)].$$

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h} \\
&= \lim_{h\to 0}\frac{f(a+h)(g(a+h)-g(a))}{h} \\
&\quad + \lim_{h\to 0}\frac{g(a)(f(a+h)-f(a))}{h} \\
&= \lim_{h\to 0} f(a+h)\cdot\lim_{h\to 0}\frac{(g(a+h)-g(a))}{h} \\
&\quad + g(a)\cdot\lim_{h\to 0}\frac{(f(a+h)-f(a))}{h} \\
&= f(a)g'(a)+f'(a)g(a).
\end{aligned}$$

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4) **Proof of the Reciprocal Rule:**

Assume that $f(x)$ is differentiable at $x = a$. Then

$$
\begin{aligned}
\left(\frac{1}{f}\right)'(a) &= \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \\
&= \lim_{h\to 0} \frac{f(a) - f(a+h)}{f(a+h)f(a)h} \\
&= -\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h\to 0} \frac{1}{f(a+h)f(a)} \\
&= -f'(a) \cdot \frac{1}{(f(a))^2} \qquad \text{(by continuity at } x = a\text{)} \\
&= \frac{-f'(a)}{(f(a))^2}\,.
\end{aligned}
$$

5) **Proof of the Quotient Rule:**

The proof of the Quotient Rule is a combination of the Product Rule and the Reciprocal Rule.

Power Rule for Differentiation

Note: We have seen that

$$\frac{d}{dx}(x) = 1 \qquad \text{and} \qquad \frac{d}{dx}(x^2) = 2x.$$

Using the Binomial Theorem we can show that if $n \in \mathbb{N}$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Theorem: [The Power Rule for Differentiation]

Assume that $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and $f(x) = x^\alpha$. Then $f(x)$ is differentiable and

$$f'(x) = \alpha x^{\alpha-1}$$

wherever $x^{\alpha-1}$ is defined.

Differentiating Polynomials and Rational Functions

Examples: Differentiating Polynomials and Rational Functions

1) Let $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial.

   Then $P(x)$ is always differentiable and

$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}.$$

2) Using the Quotient Rule, we see that a rational function

$$R(x) = \frac{P(x)}{Q(x)}$$

   is differentiable at any point where $Q(x) \neq 0$.

Differentiating Polynomials and Rational Functions

Example:

If

$$R(x) = \frac{x+2}{x^2-1},$$

then $R(x)$ is differentiable provided that $x^2 - 1 \neq 0$. That is, when $x \neq \pm 1$. Moreover,

$$\begin{aligned} R'(x) &= \frac{(\frac{d}{dx}(x+2))(x^2-1) - (x+2)(\frac{d}{dx}(x^2-1))}{(x^2-1)^2} \\ &= \frac{1 \cdot (x^2-1) - (x+2)(2x)}{(x^2-1)^2} \\ &= \frac{(x^2-1) - 2x^2 - 4x}{(x^2-1)^2} \\ &= \frac{-x^2-4x-1}{(x^2-1)^2}. \end{aligned}$$