Vertical Asymptotes and Infinite Limits

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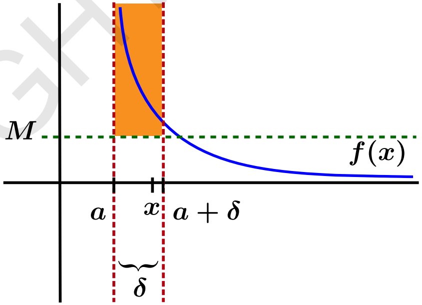
Vertical Asymptotes

Definition: [Right-Hand Infinite Limit]

We say that the limit of $f(x)$ as x approaches a from above (or from the right) is ∞, if for every $M > 0$ there exists a $\delta > 0$ such that if $a < x < a + \delta$, then

$$f(x) > M.$$

We write $\lim_{x \to a^+} f(x) = \infty$.



Vertical Asymptotes

Note: Similarly, we can define the right-hand infinite limit

$$\lim_{x \to a^+} f(x) = -\infty$$

and the two left-hand infinite limits

$$\lim_{x \to a^-} f(x) = \pm\infty.$$

Definition: [Vertical Asymptote]

We say that $x = a$ is a vertical asymptote for $f(x)$ if one of

$$\lim_{x \to a^\pm} f(x) = \pm\infty$$

holds.

Vertical Asymptotes

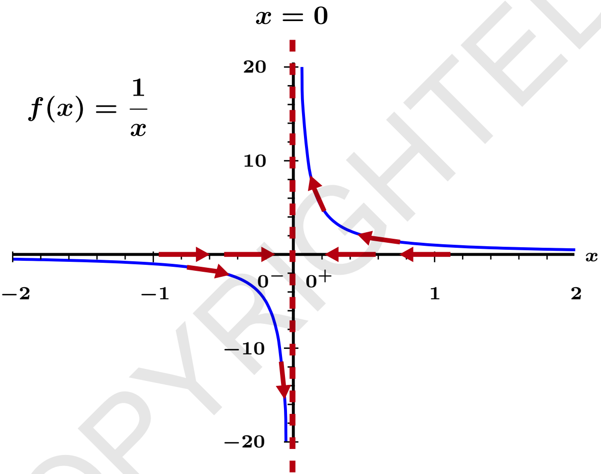
Important Note: It is important to recognize that the symbol "$\pm\infty$" is not a real number. When we say that "the limit of a function is $\pm$ infinity," we are not saying that the limit exists in the proper sense. Instead, this expression simply provides useful information about the behavior of functions whose values become arbitrarily large, either positive or negative.

When we write expressions such as

$$\lim_{x \to a} f(x) = \infty,$$

we do ***not*** mean to imply that the function $f(x)$ has a limit at the point $x = a$. Instead, this expression actually tells us that the limit of the function ***does not exist*** precisely because the function ***grows without bounds*** near a. A similar statement can be made for all of the other cases. This is a subtle point but one of which you must be aware.

Vertical Asymptotes



Example: The function $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$ since

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Vertical Asymptotes and Rational Functions

Fact: Let

$$f(x) = \frac{p(x)}{q(x)}$$

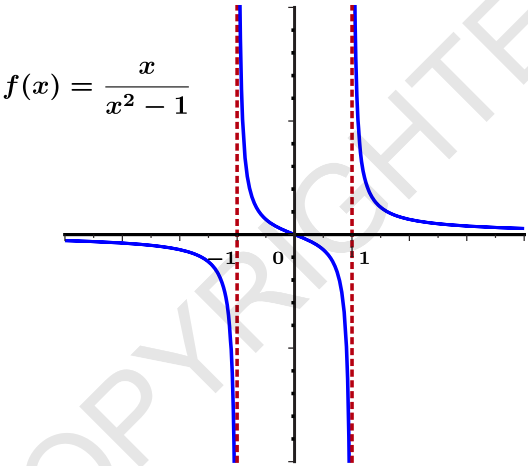
be a rational function. Then $f(x)$ has a vertical asymptote at $x = a$ if and only if $\lim_{x \to a} f(x)$ *does not exist*.

In particular, if

$$q(a) = 0 \quad \text{and} \quad p(a) \neq 0,$$

then $f(x)$ has a vertical asymptote at $x = a$.

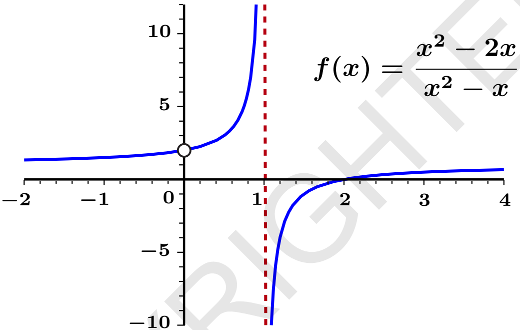
Vertical Asymptotes and Rational Functions



Example:

The function $f(x) = \frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)}$ has vertical asymptotes at $x = -1$ and $x = 1$.

Vertical Asymptotes and Rational Functions

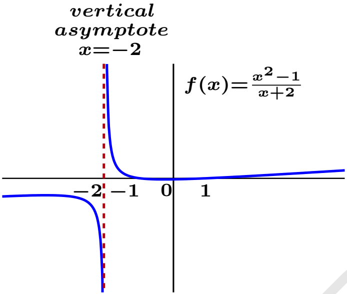


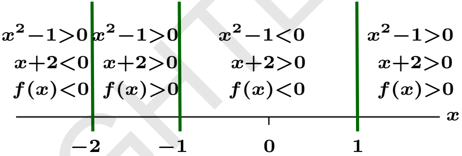
Example: Consider

$$f(x) = \frac{x^2 - 2x}{x^2 - x} = \frac{x(x-2)}{x(x-1)}.$$

Then $q(x) = x^2 - x = 0$ if $x = 0$ or $x = 1$, but only $x = 1$ is a vertical asymptote since $\lim\limits_{x \to 0} f(x)$ exists.

Vertical Asymptotes and Rational Functions





Example: Show that

$$\lim_{x\to-2^+}\frac{x^2-1}{x+2}=\infty \quad \text{and} \quad \lim_{x\to-2^-}\frac{x^2-1}{x+2}=-\infty.$$

Analyze the signs of x^2-1 and $x+2$.