Limits of Functions

Created by

Barbara Forrest and Brian Forrest

Proving the Limit Does Not Exist

Example: Show that

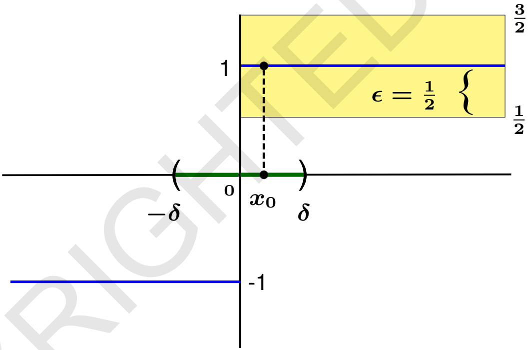
$$\lim_{x \to 0} \frac{|\,x\,|}{x}$$

does not exist.

Assume $\lim_{x \to 0} \frac{|x|}{x} = L$.

Let $\epsilon = \frac{1}{2}$.

Choose $\delta > 0$ so that if $0 <|\,x\,|< \delta$, then $|\,\frac{|x|}{x} - L\,|< \frac{1}{2}$.



Pick $0 < x_0 < \delta$. Then $\frac{|x_0|}{x_0} = 1$. Hence

$$|\,\frac{|\,x_0\,|}{x_0} - L\,|=|\,1 - L\,|< \frac{1}{2}$$

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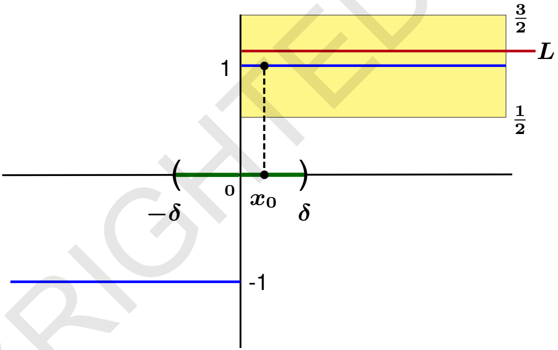

Pick $0 < x_0 < \delta$. Then $\frac{|x_0|}{x_0} = 1$. Hence

$$| \frac{|\, x_0 \,|}{x_0} - L | = | 1 - L | < \frac{1}{2} \Rightarrow \frac{1}{2} < L < \frac{3}{2}.$$

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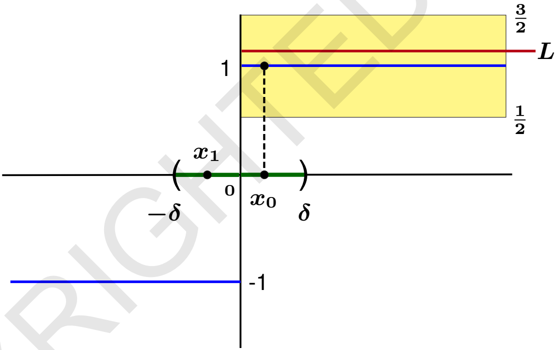

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$$| \frac{|\, x_0 \,|}{x_0} - L |=| 1 - L |< \frac{1}{2} \Rightarrow \frac{1}{2} < L < \frac{3}{2}.$$

Pick $-\delta < x_1 < 0$.

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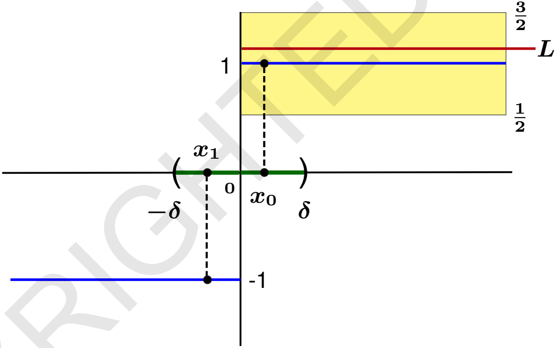

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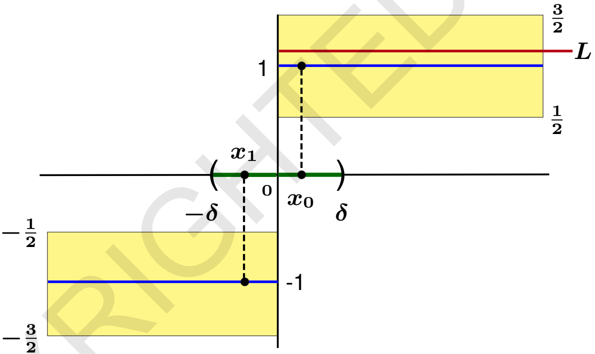

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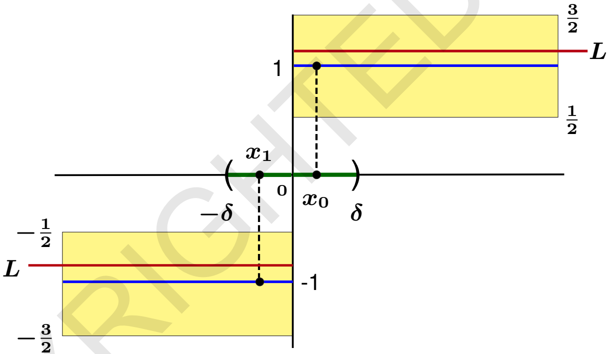

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$$| \frac{|\, x_1 \,|}{x_1} - L |=| -1 - L |< \frac{1}{2} \Rightarrow -\frac{3}{2} < L < -\frac{1}{2}.$$

Uniqueness of Limits

Exercise: Modify the argument in the previous example to prove:

Theorem: [Uniqueness of Limits]

Assume that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$. Then $L = M$.