# Squeeze Theorem for Limits of Functions

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# Squeeze Theorem for Limits

**Theorem: [Squeeze Theorem for Limits]**

Assume that $f(x)$, $g(x)$, and $h(x)$ are such that

$$g(x) \leq f(x) \leq h(x)$$

for all $x \neq a$.

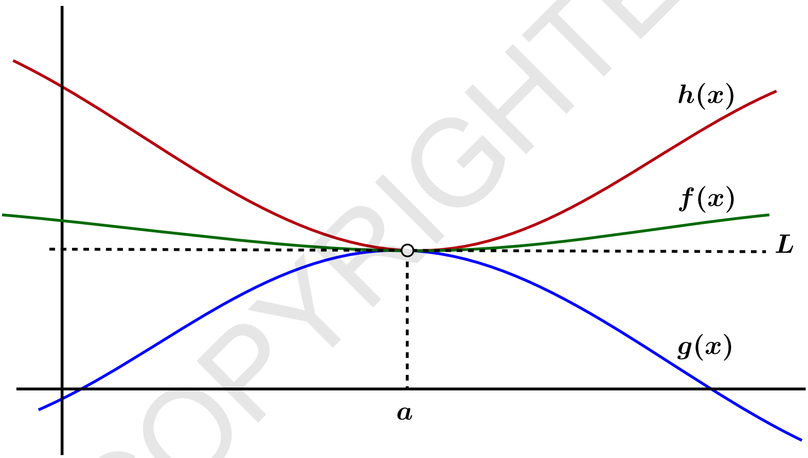
Assume also that

$$\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x).$$

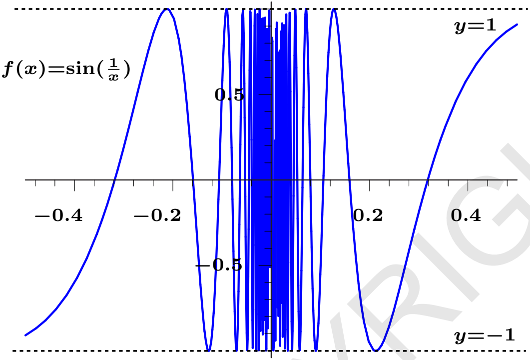
Then $\lim_{x \to a} f(x)$ exists and

$$\lim_{x \to a} f(x) = L.$$

# Squeeze Theorem for Limits



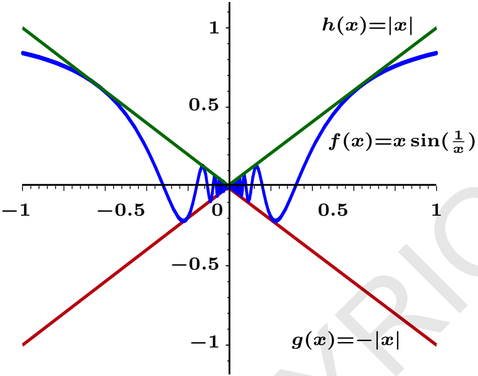
# Example




Let $f(x) = \sin(1/x)$.

We know that $\lim_{x\to 0} f(x)$ does not exist.

# Example




Let $f(x) = \sin(1/x)$.

We know that $\lim_{x\to 0} f(x)$ does not exist.

**Question:**
What can we say about $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$?

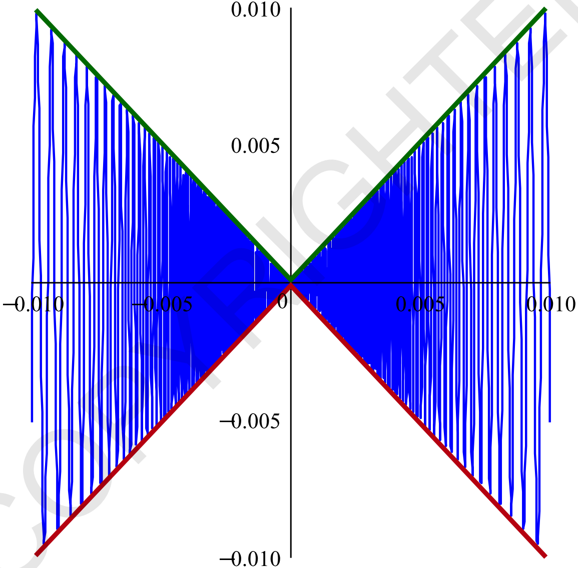
We know

$$|\sin\left(\frac{1}{x}\right)| \leq 1 \Rightarrow |x\sin\left(\frac{1}{x}\right)| \leq |x| \Rightarrow -|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|,$$

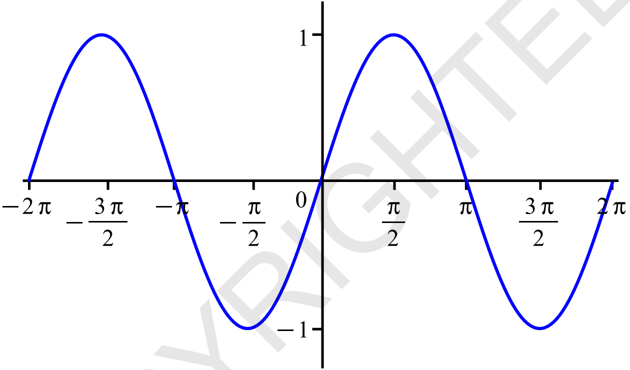
for all $x \neq 0$. Hence,

$$\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x| \Rightarrow \lim_{x\to 0} x\sin(\frac{1}{x}) = 0.$$

# Example



# $\lim_{\theta\to 0} \sin(\theta) = 0$



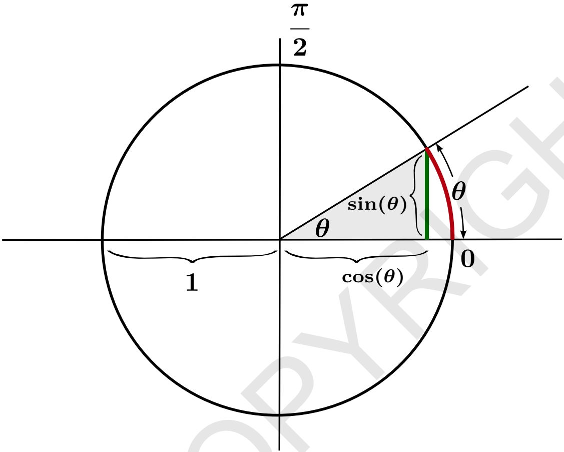



**Remark:** The graph of $\sin(\theta)$ suggests that

$$\lim_{\theta\to 0} \sin(\theta) = 0.$$

The Squeeze Theorem can be used to confirm this limit.

# $\lim_{\theta\to 0}\sin(\theta) = 0$






**Observation:**

For

$$0 < \theta \le \frac{\pi}{2},$$

we have

$$0 < \sin(\theta) < \theta.$$

Since

$$\lim_{\theta\to 0^+} 0 = 0 = \lim_{\theta\to 0^+} \theta$$

then

$$\lim_{\theta\to 0^+} \sin(\theta) = 0.$$

## $\lim_{\theta\to 0}\sin(\theta) = 0$

**Observation (continued):** Let $\frac{-\pi}{2} < \theta < 0$. Since $\sin(\theta)$ is an *odd* function,

$$\sin(-\theta) = -\sin(\theta),$$

and since if $\theta \to 0^-$, we have $0 < -\theta < \frac{\pi}{2}$ and $-\theta \to 0^+$,

$$\begin{aligned}
\lim_{\theta\to 0^-} \sin(\theta) &= \lim_{\theta\to 0^-} -\sin(-\theta) \\
&= \lim_{(-\theta)\to 0^+} -\sin(-\theta) \\
&= -\lim_{(-\theta)\to 0^+} \sin(-\theta) \\
&= -1\cdot 0 \\
&= 0.
\end{aligned}$$

Finally,

$$\lim_{\theta\to 0^-} \sin(\theta) = 0 = \lim_{\theta\to 0^+} \sin(\theta)$$

so

$$\lim_{\theta\to 0} \sin(\theta) = 0.$$

# $\lim_{\theta \to 0} \cos(\theta) = 1$

**Note:** We have just seen that

$$\lim_{\theta \to 0} \sin(\theta) = 0.$$

We also know that on $[\frac{-\pi}{2}, \frac{\pi}{2}]$,

$$\cos(\theta) = \sqrt{1 - \sin^2(\theta)}.$$

Hence,

$$\begin{aligned}
\lim_{\theta \to 0} \cos(\theta) &= \lim_{\theta \to 0} \sqrt{1 - \sin^2(\theta)} \\
&= \sqrt{1 - \lim_{\theta \to 0} \sin^2(\theta)} \\
&= \sqrt{1} \\
&= 1.
\end{aligned}$$