Sequential Characterization of the Limit

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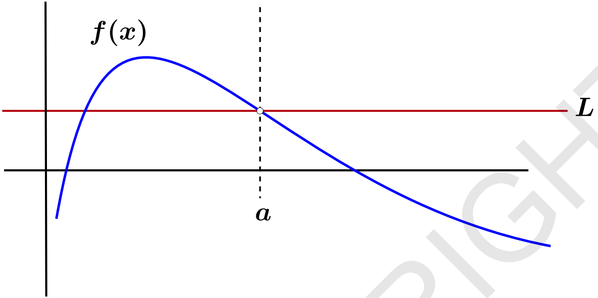
Sequential Characterization of the Limit



Assume that
$\lim_{x \to a} f(x) = L$.

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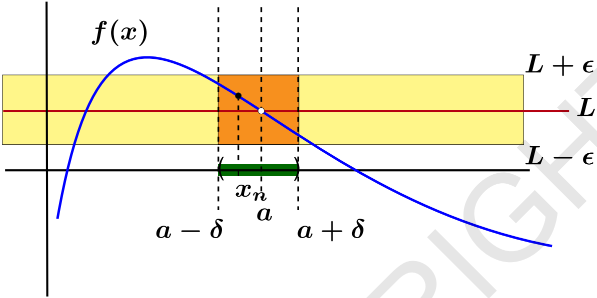


Assume that $\lim_{x\to a} f(x) = L$.

If $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < | x - a | < \delta,$$

then

$$| f(x) - L | < \epsilon.$$

Question: Can we characterize limits of functions in terms of limits of sequences? In particular, if $\{x_n\}$ is a sequence with $x_n \neq a$ and $x_n \to a$, does $f(x_n) \to L$?

Choose $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then

$$0 < | x_n - a | < \delta$$

and hence,

$$| f(x_n) - L | < \epsilon \Rightarrow \lim_{n\to\infty} f(x_n) = L.$$

Sequential Characterization of the Limit

Theorem: [Sequential Characterization of Limits]

Let $f(x)$ be defined on an open interval containing $x = a$, except possibly at $x = a$. Then the following are equivalent:

1. $\lim_{x \to a} f(x) = L$.
2. If $\{x_n\}$ is a sequence with $x_n \to a$ and $x_n \neq a$ for each $n \in \mathbb{N}$, then

$$\lim_{n \to \infty} f(x_n) = L.$$

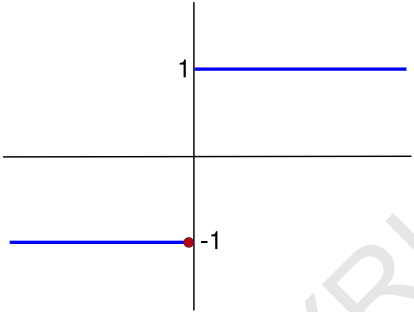
Sequential Characterization of the Limit

Theorem: [Uniqueness of Limits]

Assume that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$. Then $L = M$.

That is, the limit of a function is unique.

Example



Example:

Show that

$$\lim_{x \to 0} \frac{|\, x \,|}{x}$$

does not exist.

Assume that $\lim_{x \to 0} \frac{|x|}{x} = L$.

Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ and $f(x_n) = 1 \to 1$. So $L = 1$.

Let $y_n = \frac{-1}{n}$. Then $y_n \to 0$ and $f(y_n) = -1 \to -1$. So $L = -1$.

Conclusion: No such L exists.

Limits that Don't Exist

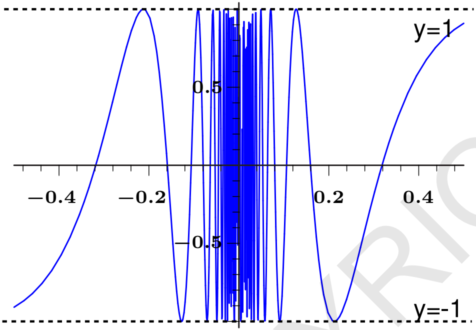
Strategy:

If you want to show that $\lim_{x \to a} f(x)$ does not exist you can do so by either of the following:

1) Find a sequence $\{x_n\}$ with $x_n \to a$, $x_n \neq a$ for which $\lim_{n \to \infty} f(x_n)$ does not exist.

2) Find two sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to a$, $x_n \neq a$ and $y_n \to a$, $y_n \neq a$ for which $\lim_{n \to \infty} f(x_n) = L$ and $\lim_{n \to \infty} f(y_n) = M$ but $L \neq M$.

A Strange Function

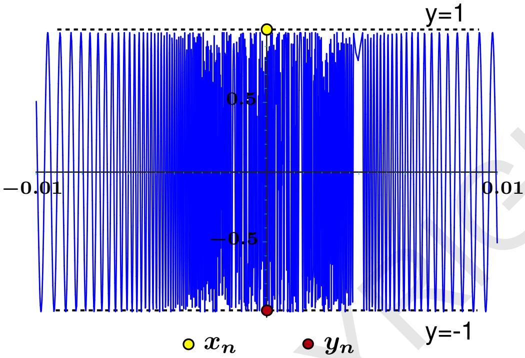


Example:

Let $f(x) = \sin(1/x)$.

What can we say about $\lim_{x \to 0} f(x)$?

A Strange Function



Example:

Let $f(x) = \sin(1/x)$.

What can we say about $\lim_{x\to 0} f(x)$?

Let

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \to 0$$

$$y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi} \to 0.$$

$$\Rightarrow f(x_n) = \sin(\tfrac{\pi}{2} + 2n\pi) = 1,\ f(y_n) = \sin(\tfrac{3\pi}{2} + 2n\pi) = -1.$$

Then

$$f(x_n) \to 1 \qquad \text{and} \qquad f(y_n) \to -1.$$

Hence $\lim_{x\to 0} f(x)$ does **not** exist.